An Invitation to Hyperbolic Geometry: gluing of tetrahedra and representation variety

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Nada se edifica sobre la piedra, todo sobre la arena, pero nuestro deber es edificar como si fuera piedra la arena

J.L.Borges, *Fragmentos de un evangelio apócrifo*

Preface

This work follows - hopefully somewhat coherently - a small journey through hyperbolic geometry and what I found most interesting in it. It is intended as a spyglass to look at some aspects of the subject for a student who finds himself interested, even only in knowing what's it about.

Hyperbolic geometry is a subject barely mentioned at undergraduate level, and rarely studied in general first- or second-year graduate courses. This comes quite surprisingly since the sheer simplicity of the geometric intuition - one of two possible negations of Euclid's parallel postulate - was first formalized by Gauss with the notion of curvature around 200 years ago.

One of the reasons is possibly that the difficulty of the questions grows rapidly, and even at a medium level the study of hyperbolic manifolds borrows tools and techniques from a vast array of subjects: differential and algebraic geometry, complex analysis, representation theory, homological algebra, just to mention the most important ones. On the other hand, this same study helps in dealing with many topological questions not directly connected to it, arising e.g. from the study of knots, and has links to theoretical physics.

Due to my background and inclination I've been more attracted to the algebraic - and at times combinatorial - aspects of the theory. This is of course reflected also in the choice of topics.

It goes without saying that a master thesis is meant to be as useful to its writer as it is to its readers. My hope is for the present work to prove itself as useful for a reader as it was for me to write it.

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Introduction

This thesis addresses graduate students without any particular background in the field: all the machinery needed will be developed without much trouble. The prerequisites are no more than the common theoretical courses of the first three years of university. That being said, a deeper acquaintance with differential and Riemannian geometry will be helpful, as will some familiarity with the theory of representations.

The first chapter is devoted not only to laying down the technical basis for the following work, but also to introduce the reader to a topic possibly new to him. I tried to do this as gently as possible, given the necessary economy of space and time: almost all the results presented there will be used or improved in the following. The majority of proofs are omitted, as they are found in any textbook on hyperbolic geometry. I tried to conserve some of the clarity of my wonderful references, to which I direct the interested reader: the first chapters of [BP12] and [Mar07] are a clear and stimulating introduction to the subject.

In the second chapter we describe the patching of hyperbolic *ideal* tetrahedra via isometries in order to obtain hyperbolic 3-manifolds. The formal idea of *side-pairing* is natural and we have found it in [Rat06].

Ideal means, basically, without vertices: this is to solve the following issue. Interesting manifold have "holes". In our situation, though, we cannot consider the natural idea of (compact) manifold with boundary. This is because every compact metric space is complete, and in the 3-dimensional case *Mostow rigidity* holds: all complete hyperbolic structures are isotopic to the identity.

To bypass this problem we consider the interior of manifolds with boundary. In order to use our knowledge of the orientation-preserving isometries of the hyperbolic 3-space $\mathcal{I}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$, we restrict to the case of M orientable, and with torus boundary. We find then a space of hyperbolic structures defined by polynomial equations in $\mathbb{C}^n_+$. We can then proceed to studying under which conditions these hyperbolic structures are complete. Following ([BP12], Section E.6) we relate this to the induced euclidean structure on the boundary. The satisfying conclusion is Proposition 2.4.4, that gives an algebraic answer to this problem too.

In the third chapter we generalize this last algebraic condition introducing the *holonomy* map. This same map turns out to be a powerful tool for describing the space of *non*-complete hyperbolic structures. In the end we point out some immediate generalizations to the case of manifolds with more than one boundary component.

The fourth chapter presents some computations around a well known example of hyperbolic 3-manifolds obtained by gluing tetrahedra. It is meant to illustrate how it is relatively easy to apply the theoretical considerations of the previous chapters; moreover, we gain some actual insight on the space of hyperbolic structures supported by the given manifold.

Notation In addition to the most common mathematical notation, which we don't recall, we will use the following symbols.
When we define something new, we will use the notation $:=$ instead of the normal equal sign.
We denote in general by k a field. We will use $\mathbb{R}_+$ for the (strictly) positive real numbers. We will frequently require a complex number to have positive imaginary part, hence the notation $\mathbb{C}_+$ for such numbers will be used. This is to avoid the use of $\mathbb{H}^2$ when not considering the hyperbolic structure, but only the underlying set. We will also write $\mathbb{C}_-$ for $-\mathbb{C}_+$.
We write $\mathbb{E}^n$ for $\mathbb{R}^n$ if we want to highlight its standard euclidean structure: for the underlying set we will still use $\mathbb{R}^n$.
If X is a smooth manifold, we denote by $\mathrm{Diff}(X)$ the group of its diffeomorphisms (under composition).
If G is any group, 1_G will denote its identity. For all groups of matrices, $\mathbb{I}_n$ is the identity n by n matrix, and we denote by $[A, B]$ the *commutator* $AB - BA$ of A and B.
S_n will denote the symmetric group on n elements.
As it is common, we will write $\mathbb{S}^n$ for the n-dimensional sphere, i.e.

$$S^n := \left\{x \in \mathbb{R}^{n+1} \colon |x| = 1\right\}$$

We will use a similar notation, $\mathbb{S}_k$, for the 2-sphere with k punctures.

Chapter 1

Preliminaries

We describe here some basic facts about the environment we will be working in. We start by recalling and collecting definitions and results over hyperbolic spaces and their groups of isometries. We will give then a description of hyperbolic ideal tetrahedra and finally introduce the developing map.

1.1 Basic notions

We recall first the notion of *completeness* for a metric space, since we will use it extensively.

Definition 1.1.1. A metric space X is *complete* if every Cauchy sequence converges to a point of X.

We give the customary definition of hyperbolic n-space, however we will usually work with a model of it, described later.

Definition 1.1.2. The *hyperbolic n-dimensional space* $\mathbb{H}^n$ is a complete, connected, simply connected real Riemannian manifold with constant sectional curvature -1.

A clear overview of various models for $\mathbb{H}^n$ can be found in Chapter A of [BP12]. We include here a brief description of the 2 and 3-dimensional *upper-half-space* model which we will use for understanding the actions of the isometry groups.

Definition 1.1.3. The algebra H of *Hamilton quaternions* is the $\mathbb{R}-$algebra generated by $1, i, j$ with the relations

$$i^2 = j^2 = -1; \quad ij = -ji$$

It contains the real algebra of complex numbers $\mathbb{C}$ generated by $1, i$, with the only relation $i^2 = -1$. We can write a general *quaternion* as $\omega = z + uj$ for $z \in \mathbb{C}$ and $u \in \mathbb{R}$.

Definition 1.1.4. The *upper-half-space* model for the hyperbolic 3-dimensional space is

$$\mathbb{H}^3 := \{z + uj \in H : z \in \mathbb{C}, u \in \mathbb{R}_+\}$$

equipped with the so-called *Poincaré metric*, given at any point $(z, u) \in \mathbb{H}^3$ by the euclidean inner product on the tangent space $T_{(z,u)}\mathbb{H}^3 \cong \mathbb{R}^3$ multiplied by u^{-2}.

It contains the *upper-half-plane* model for $\mathbb{H}^2$ as the space

$$\mathbb{H}^2 := \left\{x + uj \in \mathbb{H}^3 : x \in \mathbb{R}\right\}$$

which inherits an analogous metric.

We will usually imagine $\mathbb{H}^3$ as a subset of $\mathbb{R}^3$. The algebraic structure inherited as a subset of H is only useful, actually, to give us some sort of coordinates and to identify $\mathbb{H}^2$ in $\mathbb{H}^3$.

Remark 1.1.5. The topology induced by the Poincaré metric is the same inherited by $\mathbb{H}^3$ as a subset of $\mathbb{R}^3$. This comes from the fact that they are at every point a positive multiple one of each other, so they really look the same when zooming in enough. More formally, for every $w \in \mathbb{H}^3$ there is a small enough (open) ball B_w such that the hyperbolic distance between any two points in B_w is bounded above and below by a fixed positive multiple of the euclidean distance.

1.1.1 The ball model and the boundary

We recall that X is a locally compact, non-compact topological space, its *one-point (or Alexandroff) compactification* $\widehat{X}$ is defined as follows. Consider the underlying set

$$\widehat{X} := X \cup \{\infty\}$$

where ∞ is a point, called loosely *point at infinity* in view of its geometrical interpretation in many common examples. Then endow it with the topology given by the (inclusion of the) opens of X, plus the subsets U of $\widehat{X}$ containing $\{\infty\}$ such that $\widehat{X} \setminus U \subset X$ is closed and compact in X.

We described $\mathbb{H}^3$ as a open subset of $\mathbb{R}^3$. We write $\widehat{\mathbb{R}}^3$ for the one-point compactification of $\mathbb{R}^3$. Recall that $\widehat{\mathbb{R}}^3 \cong \mathbb{S}^4$.

We consider now $\mathbb{H}^3$ as an open subset of $\widehat{\mathbb{R}}^3$. Define $\widehat{\mathbb{H}}^3$ to be the closure of $\mathbb{H}^3$ in $\widehat{\mathbb{R}}^3$.

Since $\mathbb{H}^3$ was unbounded in $\mathbb{R}^3$, $\infty \in \widehat{\mathbb{H}}^3$. Moreover, since $\mathbb{H}^3$ is open in $\widehat{\mathbb{R}}^3$, it doesn't intersect its boundary in $\widehat{\mathbb{R}}^3$. So

$$\widehat{\mathbb{H}}^3 = \mathbb{H}^3 \cup \partial\mathbb{H}^3$$

where the boundary operation ∂ is taken in $\widehat{\mathbb{R}}^3$, and the union is disjoint.

With the notation of Hamilton quaternions, an explicit description of $\partial\mathbb{H}^3$ is straightforward. Let

$$H_0 := \{z \in H : z \in \mathbb{C}\}$$

Then

$$\partial\mathbb{H}^3 = H_0 \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$$

Similarly we have $\partial\mathbb{H}^2 \cong \mathbb{R} \cup \{\infty\} \cong \mathbb{RP}^1$. In particular, the hyperbolic 3-space is homeomorphic to a open 3-ball $\mathbb{B}^3$, and its boundary as defined above fits in as the boundary $\mathbb{S}^2 \cong \mathbb{CP}^1$ of $\mathbb{B}^3$.

This topological interpretation allows us a better understanding of the relation between the hyperbolic space and its boundary.

1.1.2 Geodesics and hyperplanes

We give a little geometric idea of these common objects in the hyperbolic setting. They will be useful to help our general visualization, but they will also be used directly.

The *geodesics* in $\mathbb{H}^3$ are either vertical lines $\{z\} \times \mathbb{R}_+$, or intersections of $\mathbb{H}^3$ with euclidean circles with centre lying in H_0 and intersecting it perpendicularly. Thus they can be identified with unordered pairs of distinct points in $\mathbb{CP}^1$, to which we will refer as "endpoints". The geodesics of $\mathbb{H}^2$ are geodesics of $\mathbb{H}^3$ lying in H^2, thus their endpoints will be in $\mathbb{RP}^1$.

The *hyperplanes* in $\mathbb{H}^3$ are either vertical half-planes, hence qualitatively similar to the inclusion in $\mathbb{H}^3$ of $\mathbb{H}^2$, or intersection of $\mathbb{H}^3$ with euclidean hemispheres with centre lying in H_0.

1.2 Groups of isometries

First, a bit of notation. For the hyperbolic n-space $\mathbb{H}^n$ we denote by $\mathcal{I}(\mathbb{H}^n)$ the group of its isometries and by $\mathcal{I}^+(\mathbb{H}^n)$ the subgroup of orientation-preserving isometries. We assume the reader is familiar with the definition of $GL_n(k)$, $SL_n(k)$ for a field k.

Recall that $PGL_n(k)$ is defined as the quotient of $GL_n(k)$ by the action of k^* given by the usual scalar multiplication of a matrix. Such a multiplication changes the determinant of a matrix by λ^n for any $\lambda \in k$. So we can define $PSL_n(k)$ as the quotient of $SL_n(k)$ by the restriction of the analogous action of $\{\lambda \in k \colon \lambda^n = 1\}$.

We describe first the isometries of $\mathbb{H}^3$, and then relate them to their "restrictions" to the boundary. Recall that in the definition we gave via the quaternion algebra H, we can think of the boundary $\partial\mathbb{H}^3$ as the point at infinity plus the horizontal plane H_0. In particular, it can be identified with $\mathbb{CP}^1$.

Lemma 1.2.1. *Then the action of $SL_2(\mathbb{C})$ on H*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .\omega = (a \cdot \omega + b)\,(c \cdot \omega + d)^{-1} \tag{1.1}$$

factors through $PSL_2(\mathbb{C})$, and the resulting action preserves $\mathbb{H}^3$.

Proof. The first statement is trivial. The computation needed for the second can be simplified by noting that $SL_2(\mathbb{C})$ is generated by *shear transformations*, i.e. matrices of the form

$$\begin{pmatrix} -1 & z \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ z & -1 \end{pmatrix}$$

with $z \in \mathbb{C}$.

Then, if $\omega \in \mathbb{H}^3$,

$$\begin{pmatrix} -1 & z \\ 0 & -1 \end{pmatrix} \omega = (-\omega + z)(-1)^{-1} = \omega - z \in \mathbb{H}^3$$

since $z \in \mathbb{C}$.

The other one is a bit more computational, but similarly easy. #

This action of $PSL_2(\mathbb{C})$ on $\mathbb{H}^3$ encodes all the orientation-preserving isometries, and an analogous statement holds for the orientation-preserving isometries of $\mathbb{H}^2$, as shown e.g. in (Theorem A.3.3, [BP12]). In particular

Remark 1.2.2.

$$\mathcal{I}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$$
$$\mathcal{I}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$$

For what concerns the orientation-reversing isometries, they are compositions of the already given orientation-preserving isometries with reflections about hyperbolic hyperplanes. We can consider without loss of generality the reflection about the hyperplane

$$\{z + uj \colon z \in \mathbb{R}\} = \mathbb{H}^2 \subset \mathbb{H}^3,$$

i.e. the conjugation map $c \colon z \longmapsto \overline{z}$; this since every hyperbolic hyperplane can be sent to $\mathbb{H}^2$ via an orientation-preserving isometry. Then we can write

$$\mathcal{I}(\mathbb{H}^3) = \mathcal{I}^+(\mathbb{H}^3) \oplus \langle c \rangle = \mathcal{I}^+(\mathbb{H}^3) \sqcup c\left(\mathcal{I}^+(\mathbb{H}^3)\right) \tag{1.2}$$

Thanks to this relation, we can keep our attention on the orientation-preserving isometries. The restriction to $\mathbb{CP}^1 \cong \partial\mathbb{H}^3 \subset H \cup \{\infty\}$ of the action (1.1) gives

$$PSL_2(\mathbb{C}) \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$$
$$\left([A], \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}\right) \mapsto \left[A \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}\right]$$

Proposition 1.2.3. *The above law defines a continuous group action. An analogous statement holds for $PSL_2(\mathbb{R})$ and $\mathbb{RP}^1$ respectively.*

Proof. We have to prove that the map is well defined, and that is an action. It is well defined because

$$\left[(\lambda A)\begin{pmatrix}\mu z_0\\ \mu z_1\end{pmatrix}\right] = \left[\lambda\mu\cdot A\begin{pmatrix}z_0\\ z_1\end{pmatrix}\right] = \left[A\begin{pmatrix}z_0\\ z_1\end{pmatrix}\right]$$

for every $\lambda, \mu \neq 0$.

We have the commutative diagram

$$\begin{tikzcd} SL_2(\mathbb{C}) \times \mathbb{C}^2 \setminus \{(0,0)\} \arrow[r] \arrow[d, "\pi \times p"] & \mathbb{C}^2 \setminus \{(0,0)\} \arrow[d, "p"] \\ PSL_2(\mathbb{C}) \times \mathbb{CP}^1 \arrow[r] & \mathbb{CP}^1 \end{tikzcd}$$

where p is the projection $\mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{CP}^1 \cong \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{C}^*$ and π is the projection $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$. The upper line is a group action, namely the restriction of the action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$. Moreover, thanks to π being a group homomorphism, the lower line, defined in order to make the diagram commute, is an action too:

$$[AB][\underline{z}] \overset{def}{=} [(AB)(\underline{z})] = [A(B(\underline{z}))] \overset{def}{=} [A][B(\underline{z})] = [A]\left([B][\underline{z}]\right)$$

The commutative diagram also implies that the action is continuous, since $\mathbb{CP}^1$ inherits the topology from $\mathbb{C}^2 \setminus \{(0,0)\}$ and the action of $SL_2(\mathbb{C})$ on $(\mathbb{C}^*)^2$ is continuous.

#

In other words, writing λ for an element of $\mathbb{CP}^1$ (resp. $\mathbb{RP}^1$), be it a real (resp. complex) number or ∞, we can write these actions as:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\lambda = \frac{a\lambda + b}{c\lambda + d}$$

for every

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in PSL_2(\mathbb{C}) \qquad (\text{resp. } PSL_2(\mathbb{R}))$$

We observe that this action is faithful: no matrix $A \in PSL_2(\mathbb{C})$ acts identically on every $\lambda \in \mathbb{CP}^1$. So we can identify an isometry of $\mathbb{H}^3$ with its associated action on the boundary; and this point of view will be very useful.

We delve now a bit deeper in the structure of the groups of isometries. We will write $GL_n^+(\mathbb{R})$ for the subgroup of $GL_n(\mathbb{R})$ consisting of matrices with positive determinant, and the analogous notion for $PGL_n(\mathbb{R})$ is:

$$PGL_n^+(\mathbb{R}) := GL_n^+(\mathbb{R})/\{\lambda \in \mathbb{R} \colon \lambda^n > 0\}$$

This means that the subgroup by which we quotient out is $\mathbb{R}^*$ if n is even and $\mathbb{R}_+$ if n is odd.

Remark 1.2.4. In particular, if n is even, $PGL_n^+(\mathbb{R})$ can really be seen as the subgroup of $PGL_n(\mathbb{R})$ containing all classes for which one (and then all) representative has positive determinant.

This is useful for us in the case $n = 2$.

Proposition 1.2.5.

$$PGL_2^+(\mathbb{R}) \cong PSL_2(\mathbb{R}), \quad PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})$$

Proof. The map

$$\begin{aligned} SL_2(\mathbb{R}) &\longrightarrow PGL_2^+(\mathbb{R}) \\ A &\longmapsto [A] \end{aligned}$$

is a group homomorphism, being composition of the inclusion $SL_2(\mathbb{R}) \hookrightarrow GL_2^+(\mathbb{R})$ and the projection $GL_2^+(\mathbb{R}) \twoheadrightarrow PGL_2^+(\mathbb{R})$.

It is surjective, because every $[A] \in PGL_2^+(\mathbb{R})$ has a positive-determinant representative, A, and then $(\sqrt{\det A})^{-1}A$ is a preimage of $[A]$ in $SL_2(\mathbb{R})$. The kernel of the above map is $\{\pm\mathbb{I}_2\}$, so we obtain

$$PGL_2^+(\mathbb{R}) \cong PSL_2(\mathbb{R})$$

The other isomorphism is checked exactly in the same way, for the complex case. #

Thanks to these isomorphisms we can consider matrices in $GL_2^+(\mathbb{R})$ and $GL_2(\mathbb{C})$, via the canonical projections, respectively as elements of $PSL_2(\mathbb{R})$ and $PSL_2(\mathbb{C})$.

We conclude this section with a more general description of hyperbolic isometries that gives maybe some geometric insight. It is found in ([BP12], Theorems A.4.2 and A.3.9 (2)).

Proposition 1.2.6. *Every isometry of $\mathbb{H}^n$ can be written as*

$$z \longmapsto \lambda \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} i(x) + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $\lambda > 0, A \in O(n-1), b \in \mathbb{R}^{n-1}$ and i is either the identity or an inversion with respect to a semisphere orthogonal to $\mathbb{R}^{n-1} \times \{0\}$. Moreover, i is the identity if and only if the isometry fixes ∞.

1.3 Hyperbolic manifolds

We define here the main properties of hyperbolic manifolds. We recall that a Riemannian metric g on a differentiable manifold M is a symmetric, positive defined 2-form on M. The couple (M, g) is referred to as a Riemannian manifold.

Definition 1.3.1. An *hyperbolic manifold* is a manifold with atlas of charts whose images are open subsets of $\mathbb{H}^n$ and transition functions are restrictions of hyperbolic isometries.

We don't ask for such a manifold to be complete, as it is commonly done; we will study later the conditions necessary to impose completeness on the manifolds we will study.

Remark 1.3.2. An equivalent way of defining such a manifold, or better such a *structure* on a manifold, is as a differentiable manifold equipped with a Riemannian metric of constant sectional curvature equal to -1. In particular, given our definition above, one gets such a Riemannian metric by pull-back of the hyperbolic metric on the open sets of the charts.

Definition 1.3.3. A *hyperbolic manifold with boundary* is the data of a manifold with boundary M and an embedding of M into an hyperbolic manifold H, of the same dimension as M. The hyperbolic metric on M is then the restriction of the hyperbolic metric of H.

Definition 1.3.4. A n-dimensional manifold M is said to be *orientable* if it admits a never-vanishing n-form. In this case, an *orientation* is the equivalence class of such a form, modulo multiplication by a function $g \in C^\infty(M, \mathbb{R}_+)$.

Example 1.3.5. $\mathbb{H}^3$ is orientable.

Proof. The form $x_1^* \wedge \ldots \wedge x_n^*$ never vanishes, being dual to the standard basis of $T_{\mathbf{x}}\mathbb{H}^n$ for each $\mathbf{x} \in \mathbb{H}^n$. #

Remark 1.3.6. We remark that, given an orientable manifold M and an embedding $N \hookrightarrow M$, an orientation is induced on N via pull-back of forms. In the previous example, the n-form given is the pull-back of the analogous "standard" n-form of $\mathbb{R}^n$ under the trivial embedding $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$.

Definition 1.3.7. Let M be an orientable Riemannian manifold. An isometry of M is said to be *orientation-preserving* if its pull back acts trivially on the orientations; *orientation-reversing* otherwise.

While this definition entails checking that the pull back of a diffeomorphism acts on the orientations, this fact can be found in a differential geometry textbook (e.g. [AT11], Section 4.2) and we omit it here.

1.4 Hyperbolic polytopes

Armed with this little arsenal we can look at the *ideal* hyperbolic triangles and tetrahedra. Ideal means their vertices lie "at infinity", e.g. an ideal triangle in $\mathbb{H}^2$ is identified by 3 distinct point on its boundary, and composed of the geodesics between them. Analogously, we aim to define an ideal tetrahedron in $\mathbb{H}^3$ in such a way that it is uniquely identified by 4 distinct, non-collinear points in $\partial\mathbb{H}^3$; in a very natural way it will inherit a own hyperbolic structure.

There are a couple of observations to make, if we want to give a good definition. First, each tetrahedron can have 2 orientations. Second, since we are interested in tetrahedra up to hyperbolic isometries, we have to identify some of the quadruples of vertices. Let we formalize this.

We write Δ^3 for the standard 3-simplex, i.e.

$$\Delta^3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \colon \sum_{i=1}^{3} t_i \leq 1, t_i \geq 0\}$$

Its *vertices* are $(0, e_1, e_2, e_3)$, and its edges are convex hulls of couples of distinct vertices.

To define properly a hyperbolic tetrahedron we need to generalize the concept of convexity to a Riemannian manifold.

Definition 1.4.1. Let M be a Riemannian manifold and $X \subseteq M$. We say that X is *geodesically convex*, or just *convex*, if for every couple of its points, there exists a geodesic arc connecting them which is contained in X.

Definition 1.4.2. A *geodesic hyperplane* in $\mathbb{H}^3$ is an isometric embedding of $\mathbb{H}^2$ into $\mathbb{H}^3$.

Because of the isometry requirement, such an embedding sends geodesics of $\mathbb{H}^2$ into geodesics of $\mathbb{H}^3$. Recall that in the hyperbolic space every two points are connected by one and only one geodesic. Therefore geodesic hyperplanes are convex.

Definition 1.4.3. An *ideal hyperbolic tetrahedron* is the image of a topological embedding

$$s \colon \Delta^3 \longrightarrow \widehat{\mathbb{H}}^3$$

such that it sends

- the vertices of Δ^3 to distinct points of $\partial\mathbb{H}^3$,
- the edges of Δ^3 to geodesics of $\mathbb{H}^3$ (plus the endpoints),
- the interior of the faces of Δ^3 to suitable subsets of geodesic hyperplanes, bounded by geodesics above.

We denote the respective images of the vertices $0, e_1, e_2, e_3$ by v_0, v_1, v_2, v_3.

It is, topologically, the same as Δ^3, and it is geodesically convex. Its non-ideal part,

$$s(\Delta^3) \cap \mathbb{H}^3$$

inherits from $\mathbb{H}^3$ a hyperbolic structure.

We consider only the image of the embedding, because we are not interested in different embeddings (complying with the above conditions) with the same image. Actually, the important remark is

Remark 1.4.4. An ideal hyperbolic tetrahedron is completely defined by the set of its vertices $\{v_0, v_1, v_2, v_3\}$, which we will call also "(non-ordered) quadruple".

The idea is that an ideal tetrahedron is a sort of "convex hull" of the 4 vertices: but we couldn't define it with such a language because the vertices themselves are not in $\mathbb{H}^3$, and $\widehat{\mathbb{H}}^3$ is not a metric space.

We haven't said anything about the orientation: with this definition tetrahedra can have both the orientation inherited from $\mathbb{H}^3$ and the opposite one. Such a choice is equivalent to choosing an ordering of the vertices, modulo even permutations.

Now, we want to give a structure to the set of oriented ideal tetrahedra up to hyperbolic isometries, and our final goal is to associate to each of them a complex number.

Definition 1.4.5. We denote by $\mathcal{R}$ the set of oriented hyperbolic ideal tetrahedra up to orientation-preserving isometry. We denote by $\mathcal{A}$ the set of ordered quadruples of distinct, non-collinear points of $\partial\mathbb{H}^3$.

Our aim is to describe $\mathcal{R}$, to give it some structure. Our first remark is that we can see it as 2 disjoint copies of a set $\mathcal{R}'$ that parametrizes the *non*-oriented hyperbolic ideal tetrahedra.

For this reason, we start considering the non-oriented tetrahedra. We use the action of S_4, by permutations, on $\mathcal{A}$. Then $\mathcal{R}'$ is $S_4 \backslash \mathcal{A}$. Now we take into account the action of orientation-preserving isometries of $\mathbb{H}^3$ on the boundary, and in particular what it does to the vertices.

Every ordered quadruple (z_0, z_1, z_2, z_3) of distinct, non-collinear points in $\mathbb{CP}^1$ can be sent to $(\infty, 0, 1, \phi(z_3))$ via a single orientation-preserving isometry ϕ. Then $\phi(z_3) \in \mathbb{C} \setminus \mathbb{R}$.

Remark 1.4.6. It may be useful to have the explicit form for ϕ. It is

$$\phi(z) = \frac{z - z_1}{z - z_0} \cdot \frac{z_2 - z_0}{z_2 - z_1}$$

Let's consider the composition map

$$\mathcal{A} \overset{\phi(z_3)}{\longrightarrow} \mathbb{C} \setminus \mathbb{R} \longrightarrow D_3 \backslash (\mathbb{C} \setminus \mathbb{R}) \tag{1.3}$$

where D_3 is the so called *dihedral group*. D_3 has presentation

$$\langle a, b \mid aba = bab, a^2 = b^2 = 1 \rangle$$

and is embedded in $PSL_2(\mathbb{C})$ as the subgroup generated by

$$i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, i \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

(corresponding respectively to a and b in the previous presentation).

Proposition 1.4.7. *The composition map in (1.3) factors through $S_4\backslash\mathcal{A}$ making the following diagram commutative.*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\phi(z_3)"] \arrow[d] & \mathbb{C}\setminus\mathbb{R} \arrow[d] \\ S_4\backslash\mathcal{A} \arrow[r, "\Phi"] & D_3\backslash(\mathbb{C}\setminus\mathbb{R}) \end{tikzcd}$$

Moreover, the map Φ is a bijection.

Remark 1.4.8. We will refer to an element of $S_4\backslash\mathcal{A}$ either as a tetrahedron T, or as an equivalence class of a quadruple of points, with the square brackets notation $[(z_0, z_1, z_2, z_3)]$. We have already noted (Remark 1.4.4) the equivalence of these interpretations.
We will use the square brackets notation also for the equivalence classes in $D_3\backslash(\mathbb{C}\setminus\mathbb{R})$.

Proof. The map factors through $S_4\backslash\mathcal{A}$ because the cross-ratio of 4 numbers changes as the D_3-action under permutations of the said numbers. Once we have proved this, the diagram commutes by construction. Surjectivity follows from the upper side of the square being surjective. Injectivity: we denote by $T_1 = [(z_0, z_1, z_2, z_3)], T_2 = [(w_0, w_1, w_2, w_3)] \in S_4\backslash\mathcal{A}$ two hyperbolic ideal tetrahedra. Let ϕ be the orientation-preserving hyperbolic isometry taking (z_0, z_1, z_2) respectively to $(\infty, 0, 1)$, and ψ be the analogous for (w_0, w_1, w_2); so that

$$\Phi(T_1) = [\phi(z_3)] \text{ and } \Phi(T_2) = [\psi(w_3)]$$

If we impose $\Phi(T_1) = \Phi(T_2)$, then $\phi(z_3)$ and $\psi(w_3)$ are in the same D_3-orbit; thus there exists a $g \in D_3$ such that

$$z_3 = (\phi^{-1} \circ g \circ \psi)(w_3)$$

Now, elements of D_3 act on $\{\infty, 0, 1\}$ by permutations: it is trivial on the generators. So

$$(\phi^{-1} \circ g \circ \psi)(w_i) = z_{\sigma(i)} \qquad \text{for a } \sigma \in S_4$$

#

Remark 1.4.9. The group D_3 is semidirect product of $\langle a\rangle$ and $\langle ab\rangle$, respectively isomorphic to C_2 and C_3. The C_2 action exchanges $\mathbb{C}_+$ and $\mathbb{C}_-$, while the C_3 action preserves both of them. So we can visualize $D_3\backslash(\mathbb{C}\setminus\mathbb{R})$ as the quotient $\langle aba\rangle\backslash\mathbb{C}_+$.

We have now found a way to associate to each ideal hyperbolic tetrahedron T with vertices (z_0, z_1, z_2, z_3) *an equivalence class of complex numbers with positive imaginary part*. We follow the notation above and call it $\Phi((z_0, z_1, z_2, z_3))$, more briefly $\Phi(T)$. We will also loosely use one of the representatives to refer to it.

1.4.1 The modulus

We go now further: we want to link to *every edge* of a tetrahedron a complex number with positive imaginary part, that we will call the *modulus* of the tetrahedron with respect to that edge. We will denote a generic edge by e or by a couple (x, y) of vertices, disregarding these latter's order.

In the above setting, we define for the tetrahedron T with vertices z_0, z_1, z_2, z_3,

$$mod(T, (z_0, z_1)) = \frac{z_3 - z_1}{z_3 - z_0} \cdot \frac{z_2 - z_0}{z_2 - z_1} \tag{1.4}$$

and the result is required to have positive imaginary part: this can be attained possibly by inverting the order of the two vertices making up the edge.

For the other edges, the formula is the same after a suitable even permutation of $z_1, \ldots, z_4$ that takes to the last two positions the vertices of the edge in question.

We will use the following

Remark 1.4.10. Let P be an ideal tetrahedron. Then we have

$$\prod_e mod(P, e) = 1 \tag{1.5}$$

where the product is taken over the edges e of P. Indeed, the 6 edges of the tetrahedron give 2 copies of the triple

$$\left\{ z, \frac{1}{1-z}, 1 - \frac{1}{z} \right\}$$

for a suitable complex number z; and the product of the three numbers is easily seen to be -1.

1.5 The developing map

We will need at least the definition of an important tool in hyperbolic geometry. We have no possibility of being complete, and refer the reader to ([Rat06], §8.4). We are interested mainly in the 2 and 3-dimensional case since we'll need the *developing map* for the study of the boundary of 3-manifolds. So, we state the following theorem for an hyperbolic manifold M of dimension 3.

Theorem 1.5.1. *Let $U \subset M$ be open, simply connected, $\varphi\colon U \longrightarrow \mathbb{H}^3$ be a chart for the hyperbolic structure. Let $\widetilde{M} \xrightarrow{\pi} M$ be the universal covering. Then the map*

$$\varphi \circ \pi\colon \pi^{-1}(U) \longrightarrow \mathbb{H}^3$$

extends to a local isometry

$$\mathcal{D}\colon \widetilde{M} \longrightarrow \mathbb{H}^3$$

unique up to composition with an element of $\mathcal{I}(\mathbb{H}^3)$. This map is called the developing map *associated to the hyperbolic structure.*

Remark 1.5.2. In particular, if M is simply connected, the developing map is a local isometry $\mathcal{D}\colon M \longrightarrow \mathbb{H}^3$.

An explicit construction is carried on in the 3-dimensional case in ([Cha], 4.0.7). In the same section is showed that *the developing map is defined up to hyperbolic isometry*.

The developing map gives rise to another idea of completeness for hyperbolic manifolds: we would like to say that such a manifold is complete if the associated developing map covers the whole of $\mathbb{H}^3$. A general description, for (X, G)-manifolds, is given in ([BP12], Section B.1), and we'll explore it in our discussion of completeness, in 2.3.

Chapter 2

Gluing tetrahedra

We describe now how to glue hyperbolic ideal tetrahedra in order to obtain more complicated hyperbolic 3-manifolds.

Since vertices of ideal tetrahedra are not in the hyperbolic space, after the gluing we expect some "holes" in the resulting manifold, usually referred to as cusps. By expanding these points we can create a boundary, without changing the topology of the manifold.

We will be interested in the case of all the vertices gluing to the same point. In this case, the boundary obtained will be homeomorphic to a 2-torus $\mathbb{T}^2$.

2.1 Construction of a manifold M with torus boundary

Definition 2.1.1. A *horosphere* centred at $p \in \mathbb{H}^3$ is a hypersurface orthogonal to all geodesics ending in p.

For every given p the horospheres centred in p are parametrized by their *radius* $r \in \mathbb{R}_+$. Despite their name, horospheres are homeomorphic to a plane, rather than a sphere: this is because the center, which is an ideal point, remains punctured.

In the upper-half-space model they are euclidean 2-spheres (of radius r) tangent to $p \in \mathbb{C}$, or horizontal planes $\{z + uj \in H : z \in \mathbb{C}, u > \frac{1}{r}\}$ when $p = \infty$.

Proposition 2.1.2. *The hyperbolic structure of* $\mathbb{H}^3$ *induces on the horospheres the euclidean structure of* $\mathbb{R}^2$.

Proof. Without any loss of generality we can consider the horosphere of radius r centred in ∞; as already noted it is a horizontal plane in $\mathbb{H}^3$ and thus the hyperbolic metric restricts on it to a positive multiple of the euclidean one, to which then it is equivalent. #

An horosphere divides the hyperbolic space in two connected components. We will call *horoball* the one whose closure in $\widehat{\mathbb{H}^3}$ intersects $\partial\mathbb{H}^3$ only in p, the center of the horosphere.

When we intersect an ideal tetrahedron with an horosphere centered at a vertex, we obtain a triangle with euclidean structure.

Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite family of disjoint ideal hyperbolic tetrahedra. We identify them with complex numbers $z_1, \dots, z_n \in \mathbb{C}_+$ as described previously.

Now we want to glue the tetrahedra on pairs of faces by hyperbolic isometries.

Definition 2.1.3. A (hyperbolic) *side-pairing* of $\mathcal{P}$ is a collection of (hyperbolic) isometries $\Phi = \{g_S\}_{S \in \mathcal{S}}$ indexed by the collection of the faces $\mathcal{S}$ satisfying, for each $S \in \mathcal{S}$, the following conditions:

1) there exists $S' \in \mathcal{S}$ such that $g_S(S) = S'$

2) $g'_S = g_S^{-1}$

3) if $S \subset P$ and $S' \subset P'$,
$$P \cap g_{S'}(P') = S$$

This last condition is to avoid overlapping of two tetrahedra which are glued on a face. This makes necessary to use orientation-reversing isometries at times, the most simple example being the Gieseking manifold.

We define an equivalence relation $\sim$ on the disjoint union of the tetrahedra $\amalg_{i=1}^n P_i$ (as a topological space) by

$$x \sim y \text{ if and only if } x = y \text{ or for some face } S, g_S(x) = y$$

Now, this is not enough. If we want the quotient topological space

$$M := (\amalg_{i=1}^n \mathcal{P}_i) / \sim$$

to be a manifold, we must prove that every point of M has a neighbourhood homeomorphic to a small euclidean ball. For points in the interior of the tetrahedra this is already clear. For points in the faces, the side-pairing only joins two faces at a time, and every face has a neighbourhood homeomorphic to a half ball. But, for points on the edges, we must impose conditions. We will do this in the next section. For now, we can just assume M is indeed a manifold.

Definition 2.1.4. An *edge* of M is the image under the quotient map of an edge of a tetrahedron $P_i \in \mathcal{P}$. Analogously, a *face* of M is the image under the quotient map of a face of a tetrahedron $P_i \in \mathcal{P}$.

Now we want to better study its boundary. If P is a hyperbolic ideal tetrahedron, we denote by $\widehat{P}$ its closure in $\widehat{\mathbb{H}}^3$, i.e. P plus its vertices. Now, with the above notation, define as $\mathcal{V}$ the set of all vertices of tetrahedra in $\mathcal{P}$. The hyperbolic isometries g_S define actions on $\partial\mathbb{H}^3$ which send some of the vertices to others.

Fix a polyhedron P. We let v vary among its vertices, and we consider pairwise disjoint horoballs centred in v. Let's denote them B_v. We take them small enough to intersect only the 3 faces of P ending in v. Clearly M is homeomorphic to the interior of the manifold with boundary $\widetilde{M}$ obtained taking out, from M, the image of these $B_v \cap P$ under the quotient map.In the following we will identify the two manifolds and refer commonly to them as M. The boundary of $\widetilde{M}$ will be then denoted by ∂M.

Remark 2.1.5 (Extension of the hyperbolic structure)**.** From the beginning we have for free a hyperbolic structure on the image of the interior of the tetrahedra.
Then, since we are working with gluing maps that are hyperbolic isometries, we can extend this structure to the interior of the gluing, i.e. to the interior of the faces.
Finally, when we will prove that M is indeed a manifold, we will prove indeed that a neighbourhood of each edge is hyperbolically isometric to a neighbourhood of a vertical geodesic in $\mathbb{H}^3$. In other words, the above hyperbolic structure extends to the edge.

2.2 Edge conditions

By (Lemma E.5.6, [BP12]) the number of edges in M is the same as the number of tetrahedra we are gluing, n. Fix one of these, call it e. We realize it in the upper-half-space model as the geodesic $(0, \infty)$.
Consider the tetrahedra containing e. Let's call them $P_1, \dots, P_m$, with possibly some repetition when two or more edges of the same tetrahedron P_i end up glued in e.
We consider their moduli $m_i := mod(P_i, e)$ with respect to their edges identified with e. Now, gluing them, let's say anti-clockwise, around the edge means getting a polygon with vertices $\infty, 0, m_1, m_1 m_2, \dots$

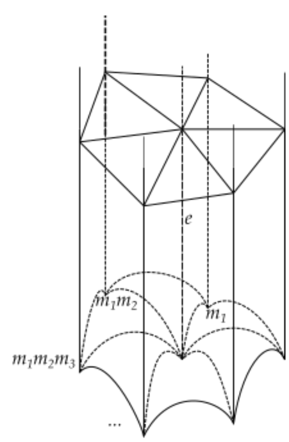

Figure 2.1: Gluing the tetrahedra $P_1, \dots, P_n$ around the edge e

We conclude that the tetrahedra glue well around e if and only if

- $\prod_{i=1}^{m} m_i = 1$ (2.1)
- $\sum_{i=1}^{m} (\arg(m_i)) = 2\pi$

If the first condition holds for all edges e then the second condition holds for all edges e, as shown in ([BP12], Lemma E.6.1).

Definition 2.2.1. We call *edge equations* the collection of (2.1) for the edges e in M.

Now we let the tetrahedra $\mathcal{P}$ and the side-pairing vary. The tetrahedra can assume all values in the space of oriented tetrahedra $\mathcal{R}$, i.e. the moduli m_i can assume all values in $\mathbb{C} \setminus \mathbb{R}$. If all the edge equations hold true, the side-pairing is said to be *proper* and the hyperbolic structure on the interior of the tetrahedra extends to the whole manifold M.

Definition 2.2.2. Let M be a manifold as above. We write $H(M)$ for the space of hyperbolic structures supported by M.

We will see $H(M)$ as a subset of $\mathbb{C}^n$, with the interpretation of ideal hyperbolic oriented tetrahedra as complex (not real) numbers we have seen in Section 1.4: elements of $H(M)$ in $\mathbb{C}^n$ are those n-ples of complex numbers corresponding to n-ples of tetrahedra that glue well around each edge. Since the conditions defining $H(M)$ are algebraic equations, it is an algebraic set. We can study then its dimension.

Remark 2.2.3. We have, at first, n algebraic equations: one for each edge. However, multiplying all of them we get, on the left-hand side,

$$\prod_{i=1}^{n} \prod_{e \in P_i} mod(P_i, e)$$

with $e \in P_i$ being the 6 edges of P_i. By Remark 1.4.10 this is 1, so the n equations are not independent.

This implies that the dimension of $H(M)$ as an algebraic set in $\mathbb{C}^n$ is at least 1.

2.3 Completeness

We turn now to investigate when the hyperbolic structure is complete.

As we said, the existence of the developing map allows another definition of completeness for a hyperbolic manifold. We refer to ([BP12], B.1) for this new definition and for the proof of its equivalence to our "metric" definition 1.1.1.

Definition 2.3.1. Let M be a hyperbolic 3-manifold. We say that M is a *complete hyperbolic manifold* if the developing map of the universal covering $\widetilde{M}$, i.e.

$$\mathcal{D} \colon \widetilde{M} \longrightarrow \mathbb{H}^3$$

is a homeomorphism.

Theorem 2.3.2. *An hyperbolic 3-manifold is complete as a Riemannian manifold if and only if it is a complete hyperbolic manifold.*

Proof. If M is complete in the metric sense, then it is isometrically isomorphic to the quotient of $\mathbb{H}^3$ by $\pi_1(M)$, identified with a convenient discrete subgroup of isometries of $\mathbb{H}^3$ ([BP12], Theorem B.1.7). In particular $\widetilde{M} = \mathbb{H}^3$ and the developing map $\mathcal{D}$ is an isometry of $\mathbb{H}^3$, thus a homeomorphism.

Vice versa, let $\mathcal{D}\colon \widetilde{M} \longrightarrow \mathbb{H}^3$ be a homeomorphism. We denote by

$$\pi\colon \widetilde{M} \longrightarrow M \cong \widetilde{M}/\pi_1(M)$$

the usual covering map. Recall that the action of $\pi_1(M)$ on $\widetilde{M}$ is free and properly discontinuous. By definition of developing map, $\mathcal{D}$ transports the hyperbolic structure from $\mathbb{H}^3$ to $\widetilde{M}$. So $(\pi \circ \mathcal{D}^{-1})(\mathbb{H}^3) \cong M$ is the quotient of $\mathbb{H}^3$ by an identification of $\pi_1(M)$ as a discrete subgroup of isometries. #

In the following we can thus use interchangeably these notions of completeness. However, before going back to our study of completeness, we give an example in dimension 2, in order to better show the behaviour of non-complete structures.

Example 2.3.3 (Complete and non-complete hyperbolic structures on $D^2 \setminus \{0\}$)**.** This example comes from ([Bon], Section 6.7) and ([BP12], Example B.1.16).

Let $\widehat{D} := D^2 \setminus \{0\}$ be the unit disk punctured in 0. We didn't introduce the *disk model* for the hyperbolic 2-space, so the reader may not know it. However, it is enough to know that D^2 can be equipped with a (complete) hyperbolic structure such that is isometrically diffeomorphic to $\mathbb{H}^2$ via an isometric diffeomorphism φ.

Now, a hyperbolic structure on $\widehat{D}$ is clearly inherited from the hyperbolic structure of D^2. Equally clear is the fact that this inherited structure is not complete.

In this case, the developing map is

$$\mathcal{D}\colon \mathbb{H}^2 \xrightarrow{\pi} \widehat{D} \hookrightarrow D^2 \xrightarrow{\varphi} \mathbb{H}^2$$

i.e. the composition of the covering map

$$\begin{aligned} \pi\colon \mathbb{H}^2 &\longrightarrow \widehat{D} \\ z &\longmapsto e^{2\pi i z} \end{aligned}$$

with the inclusion $\widehat{D} \hookrightarrow D^2$ (which is the map giving the hyperbolic structure).

We can actually ignore the last map φ, since it changes nothing from the point of view of hyperbolic metric. We have included it only for consistency with the notation of our definition of developing map.

The developing map $\mathcal{D}$ is not an homeomorphism: in fact nothing is sent to $0 \in D^2$, and then by φ to $1 \in \mathbb{H}^2$.

However, the covering map π provides and example of a complete hyperbolic structure on $\widehat{D}$. In fact, it allows and identificaton of $\widehat{D}$ with $\mathbb{H}^2/\mathbb{Z}$, under the action $z \mapsto z+n$. In other words, we can identify $\widehat{D}$ with a vertical strip $I = \{z\colon \Im(z) \in [0,1]\} \subset \mathbb{H}^2$ modulo the gluing on the edges. Since the action is free and properly discontinuous, the hyperbolic structure on the quotient space is complete.

This can be visualized by the fact that the orbit of the strip I under the action of $\mathbb{Z}$ covers all $\mathbb{H}^2$, with intersections only on the edges.

Now, let's go back to our original setting, with M a manifold glued as in Section 2.1, recall that $\partial M \cong \mathbb{T}^2$. The triangulation $\mathcal{T}$ of M in tetrahedra induces a triangulation $\mathcal{T}'$ on the boundary. By Proposition 2.1.2 each of the triangles in $\mathcal{T}'$ has a euclidean structure. Moreover, the transition maps on the sides of the triangles come from restrictions of hyperbolic isometries sending ∞ to ∞. We can describe them using Proposition 1.2.6.

Remark 2.3.4. An hyperbolic isometry fixing ∞ is of the form

$$\varphi\begin{pmatrix} z \\ h_0 \end{pmatrix} = \begin{pmatrix} \lambda A z + w \\ \lambda h_0 \end{pmatrix} \tag{2.2}$$

with $A \in O_2(\mathbb{R})$. The rule

$$z \longmapsto \lambda A z + w$$

on $\mathbb{C} \cong \mathbb{R}^2$ is a composition of dilations (multiplication by λ), rotations and reflections (action of $O_2(\mathbb{R})$) and translations (by w), it is in other words a euclidean *similarity* in that it preserves angles but not lengths.

Remark 2.3.5. Euclidean similarities form a subgroup of $\mathrm{Diff}(\mathbb{R}^n)$, and of course they include euclidean isometries $\mathcal{I}(\mathbb{E}^n)$. It might prove useful to give them their own notation: $\mathcal{S}(\mathbb{E}^n)$.

On the one hand an euclidean structure on a manifold is the data of an atlas of euclidean isometries with isometric transition maps; this gives a Riemannian structure on M. On the other hand a similarity structure is the data of an atlas of eulidean isometries with similarities - a bigger class of maps - as transition maps. This is not enough to define a Riemannian structure on the manifold.

We will mainly play with two important results. We state the first (see [BP12], Proposition E.6.5).

Proposition 2.3.6. *The hyperbolic structure on M determined by a proper side-pairing is complete if and only if the similarity structure induced on ∂M is euclidean.*

There is a nice geometric interpretation of this. We call *horizontal* the hyperbolic isometries that preserve heights. They necessarily fix ∞.

In general, the isometries of the face-pairing are not horizontal. When we cut the glued tetrahedra like in Figure 2.1 at a height h, we obtain on each triangle a euclidean structure coming from different heights. Remember, the euclidean structure is fully determined by the height of the horizontal plane in the upper-half-plane. So, they are not the same, they are related by similarities.

To ask that they patch together to a whole euclidean structure means asking of these similarities to be actually euclidean isometries. By the above Remark 2.3.4 this amounts to λ being 1 in (2.2), i.e. the isometries being horizontal.

For the proof of the above proposition we will need the second important result, but first some preparation. We have considered till now gluing of tetrahedra. When we glue different hyperbolic tetrahedra we can imagine of doing so in successive steps; in particular gluing first all the tetrahedra in a big polyhedron, and then proceeding to the gluing of its faces. This results in an analogous theory, but with only one geometric object and a less bulky side-pairing. This point of view is held in the next formulation of Poincaré's Theorem 2.3.11, drawn from ([Rat06], §11.2).

Definition 2.3.7. A family $\mathcal{A}$ of subsets of a topological space X is said *locally finite* if every $x \in X$ admits a neighbourhood intersecting a finite number of elements of $\mathcal{A}$.

A *discrete group of isometries* is simply a subgroup of $\mathcal{I}(\mathbb{H}^n)$ that from it inherits the discrete topology. We recall that

Definition 2.3.8. The action of a group G on a topological space X is *discontinuous* if for every compact subset K of X, the set $K \cap gK$ is non-empty for finitely many $g \in G$. An action is *free* if $g.x = x$ for some $x \in X$ implies that g is the identity.

Proposition 2.3.9. *A group Γ of hyperbolic isometries is discrete if and only if it acts discontinuously on $\mathbb{H}^n$.*

Proof. Direct consequence of ([Rat06], Theorem 5.3.5). #

Definition 2.3.10. A *fundamental polyhedron* for a discrete group of hyperbolic isometries Γ is a connected polyhedron P such that:

$$1) \qquad \mathbb{H}^n = \bigcup_{g \in \Gamma} gP$$

$$2) \qquad \left\{ g\mathring{P} \right\}_{g \in \Gamma} \qquad \text{are pairwise disjoint and locally finite}$$

A fundamental polyhedron is said to be *exact* if for each side S of P there is an element $g \in \Gamma$ such that

$$S = P \cap gP$$

From the setting of the previous sections - a set $\mathcal{P}$ of tetrahedra, and a side-pairing for them - we can obtain a single ideal hyperbolic polyhedron with a side-pairing, which is nothing more than a bunch of identification of its faces by hyperbolic isometries. Some edges of this polyhedron are identified by these isometries. This translates the edge equations of the tetrahedra in conditions on the composition of some face-pairings, that describe however the analogue situation. We follow ([Rat06]) in calling these conditions *cycle relations*.

And here is

Theorem 2.3.11 (Poincaré's fundamental polyhedron theorem). *Let P be a hyperbolic ideal) polyhedron with a proper hyperbolic side-pairing Φ such that the induced hyperbolic structure on the manifold M obtained by gluing together the sides of P by Φ is complete. Then the group $\Gamma \subseteq \mathcal{I}(\mathbb{H}^3)$ generated by Φ is discrete and acts freely on $\mathbb{H}^3$, P is an exact fundamental polyhedron for Γ and there is an isometry*

$$M \cong \mathbb{H}^3/\Gamma$$

A presentation for Γ is given in the following way:

- *The generators are $\{g_S \colon S \text{ is a face of } P\}$.*
- *The relations are*

$$g_S g_{S'} = 1; g_{S_1} \dots g_{S_n} = 1 \text{ (cycle relations)}$$

We can now prove Proposition 2.3.6.

Proof of Proposition 2.3.6. ($\Rightarrow$) Assume that the hyperbolic structure is complete. So it is a quotient of its universal cover $\mathbb{H}^3$ by a discrete group Γ. Its boundary ∂M can be lifted to a connected triangulated polygon; since every triangle of the triangulation has, in its interior, an euclidean metric, it will necessarily lie on a horizontal plane. Being the polygon connected, all of it will lie on a horizontal plane. The sides are identified by isometries in Γ. Since these isometries act horizontally, they fix ∞. Then, by Remark 2.3.4 they are euclidean similarities. Moreover, since they act horizontally, they are indeed isometries.

($\Leftarrow$) We cover M with an (obvious) compact part and the conical neighbourhood of the vertex given by the union of the horoballs centred in its π-preimages. It is enough to prove that this last set is complete. Assume that the structure of ∂M is euclidean. This means that ∂M is a quotient of a triangulated polygon by identification of its sides by euclidean isometries. Thus the set we are interested in is of the form $\partial M \times [t, \infty)$ modulo identification of the vertical faces operated by horizontal hyperbolic isometries. It follows that it is actually the quotient of the whole $\mathbb{C} \times [t, \infty)$ by a discrete subgroup of $\mathcal{I}(\mathbb{H}^3)$ whose elements keep ∞ fixed. Then it is complete.

#

We proceed now to study the algebraic conditions that provide completeness. The similarity structure on ∂M induces the holonomy morphism $\pi_1(\partial M) \longrightarrow \mathcal{S}(\mathbb{E}^2)$.

We restrict to the case of M orientable. Choosing an orientation on ∂M will allow us to play with the orientation-preserving isometries of the euclidean plane instead of with all the isometries, avoiding reflections. Since euclidean reflections always fix some points, they cannot be restrictions to the ∂M of hyperbolic isometries: they cannot come from discrete subgroups of $\mathcal{I}(\mathbb{H}^3)$. However, composition of (an odd number of) reflection with non trivial translations leave no fixed point, and we want to avoid this problem (at least for now). We have

Proposition 2.3.12. *An hyperbolic structure on M is complete if and only if the induced holonomy on the boundary is injective and consists of translations.*

Proof. ($\Rightarrow$) If the similarity structure on the boundary is euclidean, the image of the holonomy is contained in $\mathcal{I}^+(\mathbb{E}^2)$ which is generated by translations and rotations. However, since it comes from a discrete group of isometries of $\mathbb{H}^3$ it cannot fix any point of $\mathbb{R}^2$, and thus is generated by translations. Moreover the map has to be injective, otherwise we can take the image of one of the 2 generators of $\pi_1(\mathbb{T}^2)$ to be the identity, and the quotient of $\mathbb{R}^2$ by only one translation is not a torus.

($\Leftarrow$) We have on the torus boundary a similarity structure such that the holonomies of the two generators a, b of $\pi_1(\mathbb{T}^2)$ are translations. We cut the torus open along a longitude-meridian pair obtaining a simply connected open $U \subset \mathbb{T}^2$. Thus we can lift it to the universal cover $\mathbb{R}^2$ to a open quadrilateral V.

The developing map $\mathcal{D}$ sends its closure to $\mathbb{R}^2$, which is the model for the similarity structure. Then, we look at $\mathcal{D}(\overline{V})$. The interior is locally isometric to $\mathbb{R}^2$, since $\mathcal{D}$ extends the chart of V. On the other hand the images of the sides are identified by a translation, which is an euclidean isometry. Thus the structure on the torus boundary is euclidean.

#

Remark 2.3.13. We want to prove that, in the previous proposition, there is no need for requesting injectivity of the holonomy.

Indeed, if the holonomy consists of translations, the only way it can be not injective, is one of these translations being trivial. Then, taking the quotien of $\mathbb{R}^2$ by only one translation means that the boundary cannot be a torus. Thus we can restate the previous proposition.

Proposition 2.3.14. *An hyperbolic structure on M is complete if and only if the induced holonomy on the boundary consists of translations.*

2.4 Computation of the holonomy

Recall that every triangle in the triangulation of ∂M comes from intersecting an ideal tetrahedron with a horosphere, and then the vertices of such triangles correspond to vertical geodesics. If Δ is such a triangle and v is a vertex of Δ, we define the modulus $mod(\Delta, v)$ to be the modulus of the only tetrahedron containing Δ with respect to the edge associated to v.

Let

$$\pi_1(\partial M) \cong \pi_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

and $\{\gamma_1, \gamma_2\}$ be a set of generators for $\mathbb{Z}\oplus\mathbb{Z}$: we will loosely refer to any of them as generator of $\pi_-1(\partial M)$.

As we said, our euclidean similarities are composition of dilations, rotations and translations: identifying $\mathbb{R}^2$ to $\mathbb{C}$ we can write them as

$$z \longmapsto \lambda z + \mu$$

where $\lambda \neq 0, \mu \in \mathbb{C}$, for every $z \in \mathbb{C}$.

Remark 2.4.1. In particular, we have the identification

$$\mathcal{S}(\mathbb{E}^2) \cong \mathbb{C}^* \times \mathbb{C}$$

We will call λ the *dilation* part of the map; the dilation is trivial when $\lambda = 1$. We compute the dilation part $\delta(\gamma)$ of the holonomy relative to a generator γ of $\pi_1(\partial M)$.

We can take γ to be made of consecutive sides of the triangulation $\mathcal{T}'$. Then we choose a lift to $\mathbb{H}^3$ in order to obtain a curve starting from a fixed point, consisting of a finite number of oriented, ordered straight segments. Every segment starts where the previous ends. We consider the angle comprised between two such segments, on the right side. Since we had an orientation for the segments, this is painless. This angle encloses a finite number of lifts of triangles from $\mathcal{T}'$, let's call them $\Delta_1, \dots, \Delta_n$.

Remark 2.4.2. The similarity sending every oriented segment to its follower has dilation component equal to the product of $-mod(\Delta, v)$ for each of these triangles Δ, and v the common vertex.

We say that all these triangles, for all the vertices in γ, lie *to the right* of γ. Then

Proposition 2.4.3. *With the above notations,*

$$\delta(\gamma) = (-1)^n \prod_{\Delta} mod(\Delta, v) \tag{2.3}$$

where the product is taken over all the triangles $\Delta \in \mathcal{T}'$ lying to the right of γ, with no repetitions.

Having computed the dilation, we can now restate Proposition 2.3.12 in purely algebraic terms.

Proposition 2.4.4. *Let M be a orientable 3-manifold with torus boundary. Let γ_1, γ_2 be two generators of $\pi_1(\partial M)$. An hyperbolic structure on M is complete if and only if*

$$\delta(\gamma_i) = 1 \qquad \textit{for } i = 1, 2 \tag{2.4}$$

Chapter 3

Algebraic structure

We look now for some algebraic insight on the structure of hyperbolic structures on a manifold M obtained by gluing n tetrahedra as described in the previous chapter; we relate this structure to the representation variety. We will use the representation varieties of both the manifold and its boundary as a "projecting screen" and realize the spaces of hyperbolic structures as "images" on this screen.

The main reference for this chapter is [Cha]. Even though all the main ideas are there, notation may vary sensibly since we tried to adapt it to our setting. Recall that $H(M)$ is the space of all hyperbolic structures on a given manifold M, including the non-complete ones.

3.1 Orientable case, $\partial M = \mathbb{T}^2$

If M is orientable, from previous sections we have a fairly clear idea of the situation. Let α and β be two generators of $\pi_1(\partial M) \cong \mathbb{Z}^2$. In 2.2 we described $H(M)$ as an algebraic subset of $\mathbb{C}^n_+$ defined by the *gluing equations*, so it is natural from now on to refer to hyperbolic structures on M as n-ples $\mathbf{z} = (z_1, \ldots, z_n)$ of complex numbers.

The completeness equation tells us how to find the complete structures in $H(M)$, but we also want to study better non-complete structures. Proposition 2.4.4 tells us that the dilation component of the holonomy is crucial here. We study it nearby the complete structure. In the following we will refer to the dilation component as the "holonomy" itself, and consider it as a map

$$\begin{aligned} \mathrm{Hol}\colon H(M) &\longrightarrow \mathbb{C}^* \times \mathbb{C}^* \\ \mathbf{z} &\longmapsto (\delta_{\mathbf{z}}(\alpha), \delta_{\mathbf{z}}(\beta)) \end{aligned}$$

Remark 3.1.1. An hyperbolic structure $\mathbf{z}$ is complete if and only if

$$\mathbf{z} \in \mathrm{Hol}^{-1}((1,1))$$

We will write $H_c(M) := \mathrm{Hol}^{-1}((1,1))$ for the set of complete structures, and $H^*(M) := H(M) \setminus \ker(\mathrm{Hol})$. We copy here from ([Rat06], §11.6) a corollary of the celebrated *Mostow Rigidity theorem*, which describes the complete structures.

Corollary 3.1.2. *The hyperbolic structure on a closed, connected, orientable 3-manifold is unique up to isometry homotopic to the identity.*

The hyperbolic structure on such an hyperbolic 3-manifold M with torus boundary extends a complete hyperbolic structure on the interior of M.

The map

$$\Delta\colon \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathrm{Rep}(\pi_1(\partial M), SL_2(\mathbb{C}))$$

sending (λ, μ) to the diagonal representation

$$\rho(\alpha) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}; \quad \rho(\beta) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

is almost everywhere 2-to-1. We can realize it as the quotient map

$$\frac{\mathbb{C}^* \times \mathbb{C}^*}{(\lambda, \mu) \sim (\lambda^{\pm 1}, \mu^{\pm 1})}$$

We want to find an analogous map giving a $PSL_2(\mathbb{C})$ representation instead, but we need then another identification. Geometrically, the quotient of $\mathbb{S}_1 \cong \mathbb{C}^*/(z \sim z^{-1})$ by the multiplicative action of $\{\pm 1\}$ is a closed disc with a hole. Algebraically, the map

$$\overline{\Delta}\colon \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathrm{Rep}(\pi_1(\partial M), PSL_2(\mathbb{C}))$$

such that

$$\overline{\Delta}(\lambda, \mu)(\alpha) = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}; \quad \overline{\Delta}(\lambda, \mu)(\beta) = \begin{pmatrix} \mu^{1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix}$$

is well defined, because the two square roots of λ differ by a factor (-1) which is absorbed by $PSL_2(\mathbb{C})$. And, analogously to the previous case, is generally 2-to-1.

Composing these maps we get

$$H(M) \xrightarrow{\mathrm{Hol}} \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\overline{\Delta}} \mathrm{Rep}(\pi_1(\partial M), PSL_2(\mathbb{C}))$$

What we are doing is clear: from a hyperbolic structure on M we get a similarity structure on the torus boundary, which in turn induces an holonomy and thus a representation of its fundamental group. The similarity structure comes from the restriction of the hyperbolic structure, and thus the representation is naturally in $PSL_2(\mathbb{C})$.

In order to streamline notation we write from now on

$$X(M) := \mathrm{Rep}(\pi_1(M), PSL_2(\mathbb{C}))$$
$$X(\partial M) := \mathrm{Rep}(\pi_1(\partial M), PSL_2(\mathbb{C}))$$

On the other hand, we could directly look at the holonomy of the hyperbolic structure as a representation of $\pi_1(M)$. We will need the developing map introduced in 1.5.

The developing map translates the monodromy action of $\pi_1(M)$ on $\widetilde{M}$ to a representation $\rho \in \mathrm{Hom}\,(\pi(M), PSL_2(\mathbb{C}))$. Moreover, developing maps which differ by composition of hyperbolic isometries yield conjugated representations.

Theorem 3.1.3. *Let M be a hyperbolic 3-manifold with a finite ideal triangulation $\mathcal{T} = \{T_1, \dots, T_m\}$. Let $\widetilde{\mathcal{T}}$ be a lifting of $\mathcal{T}$ to $\widetilde{M}$. Let $x \in T_1 \subseteq M$ and fix a lifting $\widetilde{x} \in \widetilde{T_1}$. For any $\mathbf{z} = (z_1, \dots, z_m) \in H(M)$ the developing map $\delta_{\mathbf{z}}$ that sends $\widetilde{T_1}$ to the ideal tetrahedron with vertices $(\infty, 0, 1, z_1)$ induces a conjugacy class of representations in $PSL_2(\mathbb{C})$ and thus an element of $X(M)$. The resulting map*

$$D \colon H(M) \longrightarrow X(M)$$

is algebraic, 2-to-1 onto its image.

For the proof of this theorem and of the following remark we address the reader to ([Cha], §4).

Remark 3.1.4. It follows from the Mostow Rigidity theorem that all discrete faithful representations of $\pi_1(M)$ in $PSL_2(\mathbb{C})$ are conjugate, and their class is $D(\mathbf{z_0})$, for a(ny) complete hyperbolic structure $\mathbf{z_0}$.

Writing $X_0(M)$ for the connected component of $X(M)$ containing $D(\mathbf{z_0})$, the map D is "almost surjective" onto $X_0(M)$, in the sense that

$$X_0(M) \setminus D(H(M))$$

has dimension 0.

If $\mathbf{z} \in H(M)$, we will call $D(\mathbf{z})$ the *developing representation* of $\mathbf{z}$. From $e : \pi_1(\partial M) \to \pi_1(M)$ we get, by precomposition, $e^* : X(M) \to X(\partial M)$.

Theorem 3.1.5. *The following diagram commutes*

$$\begin{tikzcd} H^*(M) \arrow[r, "D"] \arrow[d, "\mathrm{Hol}"'] & X(M) \arrow[d, "e^*"] \\ \mathbb{C}^* \times \mathbb{C}^* \arrow[r, "\overline{\Delta}"] & X(\partial M) \end{tikzcd}$$

where the maps are the restriction of those defined above.

Proof. We have to prove the commutativity for $\mathbf{z}$ not complete. Let l and m be respectively the dilation component of the holonomy of the two generators α and β of $\pi_1(\partial M)$.

Remark 3.1.6. By hypothesis at least one of l and m is not 1. Moreover, if the other one happens to be 1 too, then the associated similarity is the identity. E.g. see ([BP12], Lemma E.6.6, (a)$\Rightarrow$(d)).

We compute $(e^* \circ D)(\mathbf{z})$, which is completely determined by $D(\mathbf{z})(e(\alpha))$ and $D(\mathbf{z})(e(\beta))$.

Recall the notation of Theorem 3.1.3. We can lift $e(\alpha)$ and $e(\beta)$ to $\widetilde{M}$ getting paths $\widetilde{e(\alpha)}$ and $\widetilde{e(\beta)}$ on a lift of ∂M. The hyperbolic structure of M induces on ∂M a similarity structure. Then the representation induced by the developing map acts on $e(\alpha)$ and $e(\beta)$ as euclidean similarities. In particular,

$$D(\mathbf{z})(e(\alpha)) = [z \longmapsto lz + z_0]$$
$$D(\mathbf{z})(e(\beta)) = [z \longmapsto mz + z_1]$$

We have the natural embedding

$$\mathcal{S}(\mathbb{E}^2) \cong \mathbb{C}^* \times \mathbb{C} \longrightarrow PSL_2(\mathbb{C})$$
$$[z \mapsto az + b] \longmapsto \pm \begin{pmatrix} \sqrt{a} & b/\sqrt{a} \\ 0 & 1/\sqrt{a} \end{pmatrix}$$

Hence we obtain $(e^* \circ D)(\mathbf{z}) \in X(\partial M)$ as the conjugacy class of the following representation

$$(e^* \circ D)(\mathbf{z})(\alpha) = \pm \begin{pmatrix} \sqrt{l} & z_0/\sqrt{l} \\ 0 & 1/\sqrt{l} \end{pmatrix}$$
$$(e^* \circ D)(\mathbf{z})(\beta) = \pm \begin{pmatrix} \sqrt{m} & z_1/\sqrt{m} \\ 0 & 1/\sqrt{m} \end{pmatrix}$$

However, there is a caveat. If this is to be a representation of $\pi(\partial M) \cong \mathbb{Z}^2$, the matrices must commute, i.e. it must hold

$$[(e^* \circ D)(\mathbf{z})(\alpha), (e^* \circ D)(\mathbf{z})(\beta)] = 0$$

Now, we walk the other side of the square.
$\mathrm{Hol}(\mathbf{z})(\alpha) = l$ and $\mathrm{Hol}(\mathbf{z})(\beta) = m$ from our definitions. Applying $\overline{\Delta}$ we obtain

$$\overline{\Delta}(\mathrm{Hol}(\mathbf{z}))(\alpha) = \pm \begin{pmatrix} \sqrt{l} & 0 \\ 0 & 1/\sqrt{l} \end{pmatrix}$$
$$\overline{\Delta}(\mathrm{Hol}(\mathbf{z}))(\beta) = \pm \begin{pmatrix} \sqrt{m} & 0 \\ 0 & 1/\sqrt{m} \end{pmatrix}$$

modulo conjugation.

If one of these matrices is the identity, wlog $l = 1$, then z_0 must be 0 by Remark 3.1.6, and then we can conjugate

$$\begin{pmatrix} \sqrt{m} & z_1/\sqrt{m} \\ 0 & 1/\sqrt{m} \end{pmatrix}$$

to

$$\begin{pmatrix} \sqrt{m} & 0 \\ 0 & 1/\sqrt{m} \end{pmatrix}$$

with no consequences on the identity matrix.

Otherwise it suffices to recall that if two diagonalizable complex matrices commute, then they are simultaneously diagonalizable. #

We will use the notation $D_{\partial M} := e^* \circ D = \overline{\Delta} \circ \mathrm{Hol}$.
When the hyperbolic structure is complete the map Hol is of no use: in fact the dilation of the holonomy is trivial. So we cannot recover from it alone the developing representation of $\pi_1(M)$ on the boundary, which consists in this case of non-trivial translations.

The problem in this case arises from the bad structure of $X(\partial M)$ around the trivial representation.

We can consider as a pseudo-inverse of $\overline{\Delta}$ the map

$$t \colon X(\partial M) \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$$

sending a representation to the square of the maximal eigenvalue of the generators. This map is invertible - and actually an isomorphism - with inverse $\overline{\Delta}$ on the *regular part* of $X(\partial M)$ and is only not injective over $(1,1) \in \mathbb{C}^*$, where it sends all representations made of translations.

3.2 Dimension of $H(M)$

In Remark 2.2.3 we observed that, when $\partial M \cong \mathbb{T}^2$, the dimension of $H(M)$ as an algebraic subset of $(\mathbb{C}_+)^n$ is at least 1.

If now we denote by $z_0 \in H_c(M)$ a complete structure, we can prove that the dimension of $H(M)$ in a neighbourhood of z_0 is actually 1. Since $H(M)$ is an algebraic set, this shows that every connected component of $H(M)$ containing a complete structure is an algebraic curve.

To this end, we need a lemma.

Lemma 3.2.1. *$H_c(M) \subseteq H(M)$ is an algebraic set.*

Proof. We have seen in Proposition 2.4.4 that $H_c(M)$ is defined, in $H(M)$, by the equations (2.3). The latter are rational equations: for every tetrahedron T and edge e, the modulus mod (T, e) is rational in $\Phi(T)$, as we have shown in Remark 1.4.10. #

Now we can prove that

Proposition 3.2.2. *$H_c(M)$ as a subset of $H(M)$ is made of isolated points.*

Proof. Let $z_0 \in H_c(M)$. We focus on a neighbourhood of z_0. Since both $H(M)$ and $H_c(M)$ are algebraic sets, if z_0 weren't isolated, then the cardinality of $H_c(M)$ would be uncountable.

We want to show this is not possible. Recall that every element of $H(M)$, in particular of $H_c(M)$, corresponds to a decomposition of M in ideal tetrahedra. Let z be in $H_c(M)$ and consider the corresponding decomposition. Since the hyperbolic structure is complete, we can realize M as $\mathbb{H}^3/\Gamma$, for a discrete - and then necessarily countable - group of isometries Γ. The edges of the tetrahedra are projections of geodesic lines of $\mathbb{H}^3$ from a to b, with $a, b \in \partial\mathbb{H}^3$. In M, they are asymptotic to cusps. By construction the cusps come from fixed points of the *parabolic* elements of Γ, i.e. elements with only 1 fixed point in $\partial\mathbb{H}^3$. But there is at most a countable number of these, hence at most a countable number of edges. So the cardinality of $H_c(M)$ is at most countable. #

We want to show now that

Theorem 3.2.3. *If M is a hyperbolic 3-manifold with $\partial M \cong \mathbb{T}^2$, then the dimension of $H(M)$ in a neighbourhood of a complete structure is 1.*

Let $\mathrm{Def}(z_0)$ denote a neighbourhood of z_0 in $H(M)$ containing no other complete structures: the *Deformation space* of M equipped with the complete structure z_0.

Proof. Lemma E.6.17 in [BP12] applied to our case $k = 1$ says that the two completeness conditions, given by the two generators of $\pi_1(\partial M)$, are equivalent on $\mathrm{Def}(z_0)$. So we can reduce to one equation. The zero-locus of this equation in $\mathrm{Def}(z_0)$ is a point. Since we are in $\mathbb{C}^2$, the (complex) dimension of $\mathrm{Def}(z_0)$ must be less or equal than 1. Thanks to Remark 2.2.3, we conclude that $\dim \mathrm{Def}(z_0) = 1$. #

Chapter 4

Some Examples

A small closed neighbourhood of a knot in $\mathbb{R}^3$ is a full torus $D_2 \times \mathbb{S}^1$, this implies that its complement - homeomorphic to the complement of the knot itself - is a 3-manifold that is the interior of a 3-manifold with torus boundary. The easy conclusion is that we can find a lot of these manifolds as complement of knots, and then study the (complete) hyperbolic structures they support.

4.1 The figure-8 knot complement

There is a nice way to obtain from any knot a triangulation of its complement in ideal tetrahedra. It is described in ([Rat06], §10.3). We take it for granted and consider the case of the figure-8 knot.

Its complement in $\mathbb{R}^3$ is homeomorphic to the 3-manifold with torus boundary obtained by gluing two tetrahedra A, B as in Figure 4.1. Let respectively z and w be the complex numbers that identify the two tetrahedra: with the language of Section 1.4, $z = \Phi(A)$ and $w = \Phi(B)$. We first want to obtain, from this triangulation $\mathcal{T} = \{A, B\}$ of the manifold, a triangulation $\mathcal{T}'$ of the boundary. Every edge of the two tetrahedra corresponds to a vertex

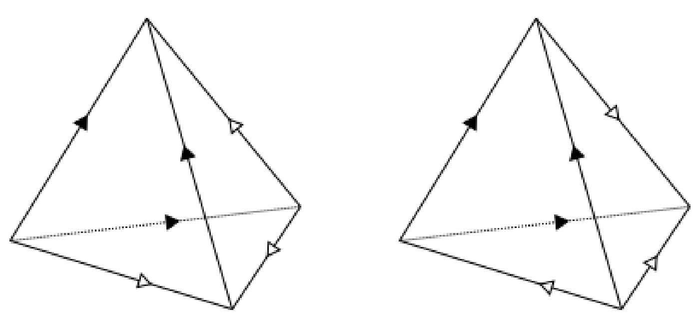

Figure 4.1: The gluing pattern for the figure-8 knot complement

of $\mathcal{T}'$. We refer to every edge/vertex by its modulus $mod(\Delta, e)$ even if this may be not so clear: what we are really interested in is only the modulus.

We draw the triangles of $\mathcal{T}'$ following the gluing of their sides induced by the gluing of the faces of A and B. We get a fundamental domain analogous to the one in Figure 4.2, which is obtained for the only z, w satisfying the completeness condition.

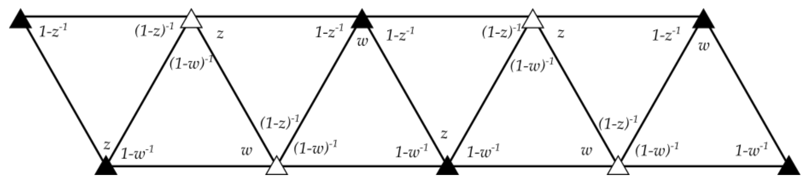

Figure 4.2: Triangulation $\mathcal{T}'$ of $\mathbb{T}^2 \cong \partial M$ for the only complete hyperbolic structure of the figure-8 knot complement, with moduli $z = w = \zeta_6$. Vertices of $\mathcal{T}'$ corresponding to the same point of ∂M are marked with the same symbol.

Gluing and completeness equations We have two gluing equations, but by Remark 2.2.3 they are not independent, so it is enough to find one. To this end we realize that the edges around which the gluing takes place are represented by the 0-cells of the triangulation $\mathcal{T}'$. Then it is sufficient to choose one of this points and to multiply the moduli around it, and to impose the result to be 1. For the one denoted by $\triangle$ in the picture, we get the equation

$$\frac{1}{(1-z)^2}\frac{1}{(1-w)^2}zw = 1 \tag{4.1}$$

For what concerns completeness, we follow Section 2.4. Let α and β be the two generators of $\pi_1(\mathbb{T}^2)$ corresponding to the "vertical" and "horizontal" sides in the above Figure 4.2. We see that the triangles lying on the right of α are the first two from the left, and the triangles laying on the right of β are all the triangles in $\mathcal{T}'$.

Remark 4.1.1. In our situation the paths α and β are very particular. The following argument, made for α, can be restated analogously for β.

For every vertex v in α there are exactly three triangles lying to the right of α which contain v. We will say also that they are *lying to the right of* v: this will cause no confusion as long as the path α we are working with is clear.
It is also worth noting that every such triangle intersects α at most in 1 edge. For each vertex v, we can order these three triangles, from 1 to 3, since we have for α both a starting vertex and a orientation. (A formal definition of this order would be a bit heavy: one can give a parametrization of α and use intersections of neighbourhoods of points of α with the triangles)

Of these three triangles, the first and the last share with α an edge touching v. Moreover, if we denote by $s(v)$ the vertex in α which comes right after v, according to α's orientation, then the last triangle to the right of v is the same that is first to the right of $s(v)$. When we multiply moduli according to (2.3) we will then find the product of two moduli for every such triangle. Recalling that the product $z(1-z)^{-1}(1-z^{-1})$ of the moduli of a triangle sums up to -1, this product equals (-1) times the inverse of the modulus with respect to the opposite vertex.

Hence for any such path γ it is sufficient to multiply alternatively, for each triangle T lying at the right of a vertex v, either $mod(T, v)$ or $-mod(T, e)^{-1}$, where e is the vertex of T opposite to the edge of γ contained in T.

The equations (2.4) become in our case of α and β:

$$\begin{aligned}
&\begin{cases} 1 = -(1-z)\left(1-w^{-1}\right) \\ 1 = -z^{-1}(1-w)^{-1}(z-1)\,w\left(-z^{-1}\right)(1-w)^{-1}(z-1)\,w \end{cases} \\
\Longleftrightarrow &\begin{cases} z - zw - 1 = 0 \\ (w-z)(w+z-2zw) = 0 \end{cases} \\
\Longleftrightarrow &\begin{cases} z - zw - 1 = 0 \\ (w-z)(w-z+2) = 0 \end{cases}
\end{aligned} \tag{4.2}$$

Complete hyperbolic structure It is easy to solve (4.2), since from the second equation we have only the two possibilities $w = z$ or $w = z - 2$. Substituting in the first equation they imply respectively

$$z^2 - z + 1 = 0$$

and

$$z^2 - 3z + 1 = 0$$

The second has real solutions, thus degenerate. Solving the first we get

$$z = w = \zeta_6 = \frac{1 + i\sqrt{3}}{2}$$

as the positive imaginary part solution. This is easily checked to satisfy (4.1), so is the only complete hyperbolic structure supported by the figure-8 knot complement.

Holonomy representation We compute explicitly the map $D_{\partial M}$ of Section 3.1 in the case of our example. We have

$$\begin{aligned}
\mathrm{Hol}(\mathbf{z}) &= (\delta_{\mathbf{z}}(\alpha), \delta_{\mathbf{z}}(\beta)) = \\
&= \left((z-1)\,\frac{w-1}{w}, \frac{(z-1)^2}{(w-1)^2} \cdot \frac{w^2}{z^2}\right)
\end{aligned}$$

by (2.3). We get then the representation

$$D_{\partial M}(\alpha) = \begin{pmatrix} \sqrt{(z-1)(w-1)w^{-1}} & 0 \\ 0 & \sqrt{(z-1)^{-1}(w-1)^{-1}w} \end{pmatrix}$$

$$D_{\partial M}(\beta) = \begin{pmatrix} (z-1)(w-1)^{-1}wz^{-1} & 0 \\ 0 & (z-1)^{-1}(w-1)w^{-1}z \end{pmatrix}$$

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