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Ordered Banach Algebras

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Contents

Introduction

In this thesis we will study ordered Banach algebras. Ordered Banach algebras are quite natural objects in analysis. Examples are algebras of spaces of functions and algebras of operators on ordered Banach spaces. Yet despite their natural occurrence, their general theory is relatively new and the results are scattered in the literature. The goal of this thesis is to give one a good insight in what is known about these functional analytic objects. We give a short introduction to the theory studied in each chapter.

In Chapter 1, which contains the preparatory material, we define an algebra cone C of a real or complex Banach algebra A. It induces on A an ordering that is compatible with the algebraic structure of A, and the pair (A, C) is then called an ordered Banach algebra (OBA). We also define some properties of C, of which normality is the most important one. The algebra cone C is said to be normal if there exists a constant $\beta \geq 1$ such that for all a, b in A with $0 \leq a \leq b$, we have that $\|a\| \leq \beta\|b\|$.

In Chapter 2 we will establish properties of the spectral radius in an OBA. The spectral radius r is said to be monotone if $0 \leq a \leq b$ implies that $r(a) \leq r(b)$. We prove that, if the algebra cone C is normal, then the spectral radius is monotone. Also we will answer the question under which conditions the spectral radius of a positive element a is contained in the spectrum $\sigma(a)$ of that element. It turns out that monotonicity of the spectral radius implies this property.

In Chapter 3 we look at poles of the resolvent function and investigate what role they play in spectral theory in OBA's. First we prove several versions of the Krein-Rutman Theorem, which is originally in terms of operators, in the context of OBA's. These theorems describe conditions under which the spectral radius of a positive element will be an eigenvalue of that element, with a positive eigenvector.

After that we look at the structure of the spectrum $\sigma(a)$ and what properties this structure forces on a. One of these properties is whether positivity of a implies that $a \geq 1$. More general, for a function f that is holomorphic on some open neighborhood of $\sigma(a)$, under what conditions of $\sigma(a)$ does $a \geq 0$ implies that $f(a) \geq 0$?

In Chapter 4 we prove several representation theorems for OBA's. In these theorems we show that an OBA satisfying certain conditions is isomorphic to the space of real-valued continuous functions $C_0(X)$ for a suitable locally compact Hausdorff space X, so that, in particular, the algebra is commutative. We will make precise which space this is.

In Chapter 5 we define the boundary spectrum. We discuss several properties of this set and investigate its relation with the spectral radius.

In Chapter 6 we turn our attention to the continuity of the spectrum and the spectral radius. If a Banach algebra A is commutative, the spectrum and spectral radius are uniformly continuous on A. If A is not commutative, this need not be the case, but in OBA's we can define subsets on which the spectral radius is always continuous.

In the last section we will prove several convergence properties for specific points in the spectrum.

In Chapter 7 we deal with domination properties related to the spectrum. That is, we provide conditions under which certain spectral properties of a positive element b will be inherited by positive elements dominated by b. Some of the results rely on subharmonic analysis.

Chapter 1

Preliminaries

1.1 Banach algebras

With $\mathbb{F}$ we will denote the field $\mathbb{R}$ or $\mathbb{C}$. Let A be a Banach algebra over $\mathbb{F}$. If A has a unit element, e, then it is assumed that $\|e\| = 1$. If A has an identity, the map $\alpha \mapsto \alpha e$ is an isomorphism of $\mathbb{F}$ into A and $\|\alpha e\| = |\alpha|$. So it will be assumed that $\mathbb{F} \subset A$ via this identification. Thus the identity will be denoted by 1 and αe just by α.

If A contains a unit element 1, we call it a unital algebra. If it does not have an unit element we can take the direct sum of A with the field $\mathbb{F}$, which gives us a unital algebra $A \oplus F$ over F:

Proposition 1.1 *If A is a Banach algebra without an identity, let $A_e = A \oplus \mathbb{F}$. Define algebraic operations on A_e by*

(i) $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$.

(ii) $\beta(a, \alpha) = (\beta a, \beta\alpha), \quad \beta \in \mathbb{F}$.

(iii) $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$.

Define $\|(a, \alpha)\| = \|a\| + |\alpha|$. Then A_e with this norm and the algebraic operations defined in (i),(ii) and (iii) is a Banach algebra with identity $(0, 1)$ and $a \mapsto (a, 0)$ is an isometric isomorphism of A into A_e.

Proof: We will only show that the given norm is compatible with the multiplication, it is easy to verify that the other Banach algebra axioms hold and $a \mapsto (a, 0)$ is an isometric isomorphism. If (a, α), $(b, \beta) \in A_e$, then $\|(a, \alpha)(b, \beta)\| = \|(ab + \beta a + \alpha b, \alpha\beta)\| = \|ab + \beta a + \alpha b\| + |\alpha\beta| \le \|a\|\|b\| + |\beta|\|a\| + |\alpha|\|b\| + |\alpha||\beta| = \|(a, \alpha)\|\|(b, \beta)\|$. □

So we can always adjoin a unit element. If A is unital, then we let $A_e = A$.

If A has an identity, we call an element $a \in A$ left invertible if there exists an element $x \in A$ such that $xa = e$. Similarly, a is right invertible if there exists an element $x \in A$ such that $ax = 1$. We call a invertible if there exists an element $x \in A$ such that $xa = ax = e$. If there are $x, y \in A$ such that $xa = e = ay$, then $y = ey = (xa)y = x(ay) = xe = x$. In particular, if a is invertible there exists a unique element a^{-1} such that $aa^{-1} = a^{-1}a = e$, called the inverse of a.

All the Banach algebras we use will be over $\mathbb{F}$, unless stated otherwise.

Lemma 1.2 *If A is a Banach algebra with identity and $x \in A$ such that $\|x - a\| < 1$, then x is invertible.*

Proof: This is Lemma 7.2.1. in [10] $\square$

Theorem 1.3 *Let A be a Banach algebra with identity, $G_l = \{a \in A : a \text{ is left invertible}\}$, $G_r = \{a \in A : a \text{ is right invertible}\}$, $G = \{a \in A : a \text{ is invertible}\}$, then G_l, G_r and G are open subsets of A. Also, the map $a \mapsto a^{-1}$ of $G \to G$ is continuous.*

Proof: This is Lemma 7.2.2. in [10] $\square$

Now we define a very important concept, the spectrum:

Definition 1.4 If A is a Banach algebra with identity over $\mathbb{F}$ and $a \in A$, the spectrum of a, denoted by $\sigma(a, A)$, is defined by

$$\sigma(a, A) = \{\alpha \in \mathbb{F} : a - \alpha \text{ is not invertible}\}.$$

The resolvent set of a is defined by $\rho(a, A) = \mathbb{F} \setminus \sigma(a, A)$.

We will omit A in above definitions if it is clear what algebra is meant, writing $\sigma(a)$ and $\rho(a)$.

Theorem 1.5 *If A is a Banach algebra over $\mathbb{C}$ with an identity, then for each a in A, $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. Moreover, if $|\alpha| > \|a\|$, $\alpha \notin \sigma(a)$.*

Proof: This is Theorem 3.6 in [10]. $\square$

From this theorem it follows that the resolvent set contains one unbounded connected component. The boundary of this component will be denoted by $\partial_\infty \sigma(a)$.

Definition 1.6 If A is a Banach algebra with identity and $a \in A$, then we define the distance $\delta : A \to \mathbb{R}_{\geq 0}$, by $\delta(a) := d(0, \sigma(a))$.

Theorem 1.7 *Let A be a Banach algebra and suppose that (a_n) is a sequence in A such that $a_n \to a \in A$. If (α_n) is a sequence such that $\alpha_n \in \sigma(a_n)$ for all $n \in N$ and $\alpha_n \to \alpha$, then $\alpha \in \sigma(a)$.*

Proof: Suppose that (α_n) is a sequence such that $\alpha_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$. Then $(a_n - \alpha_n)$ is not invertible, and $(a_n - \alpha_n) \to (a - \alpha)$. The set of invertible elements in A is open [[10] theorem 7.2.2], so the set of non-invertible elements is closed and therefore $(a - \alpha)$ is not invertible. $\square$

We will call the function $z \to (z-a)^{-1}$ from $\rho(a)$ to A the resolvent (function) of a and denoted it with $R(z, a)$.

Theorem 1.8 *The resolvent is an analytic function defined on $\rho(a)$.*

Proof: This is Theorem 7.3.8 in [10]. □

Definition 1.9 If A is a Banach algebra with identity and $\alpha \in A$, the spectral radius, $r(a)$, of a is defined by

$$r(a, A) = \sup\{|\alpha| : \alpha \in \sigma(a)\}.$$

Because σ is a nonempty and compact subset of $\mathbb{R}$ or $\mathbb{C}$, $r(a)$ is well defined, finite and the supremum is attained. From Theorem 1.5 it follows that $r(a) \leq \|a\|$.

Theorem 1.10 *If A is a Banach algebra over $\mathbb{C}$ with identity and $a \in A$, then $\lim\limits_{n\to\infty} \|a^n\|^{1/n}$ exists and*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

.

Proof: This is Theorem 3.8 in [10]. □

Definition 1.11 If A is a Banach algebra with identity and $\alpha \in A$, the peripheral spectrum, $\mathrm{psp}(a, A)$, of a is defined by

$$\mathrm{psp}(a, A) = \sigma(a, A) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(a, A)\}.$$

We introduce an important Banach algebra and state the Stone-Weierstrass Theorem.

Let X be any Hausdorff space. Let $f, g : X \to \mathbb{F}$ be continuous functions. Define the operations $(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$ and $(\alpha f)(x) = \alpha f(x)$. Define the map $\|f\| := \sup\{|f(x)| : x \in X\}$. Then the space $C_b(X)$ of all continuous functions $f : X \to \mathbb{F}$ with $\|f\| < \infty$ is a Banach algebra with the defined operations and $\|\cdot\|$ as norm. If X is locally compact, let $C_0(X)$ be the space consisting of all continuous functions $f : X \to \mathbb{F}$ such that for all $\epsilon > 0$, the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. Then $C_0(X)$ is a closed subalgebra of $C_b(X)$ and thus a Banach algebra. If X is compact, we denote with $C(X)$ the space of all continuous functions $f : X \to \mathbb{F}$, and we have $C_0(X) = C_b(X) = C(X)$.

Theorem 1.12 (The Stone-Weierstrass Theorem) *If X is a compact Hausdorff space and A is a closed subalgebra of $C(X)$ such that*

(i) $1 \in A$;

(ii) A separates the points of X;

(iii) if $f \in A$, then $\overline{f} \in A$.

Then $A = C(X)$.

Proof: This is Theorem 5.8.1 in [10]. □

From this theorem, it can be shown that

Corollary 1.13 *If X is a locally compact Hausdorff space and A is a closed subalgebra of $C_0(X)$ such that*

(i) for each x in X there is an f in A such that $f(x) \neq 0$;

(ii) A separates the points of X;

(iii) if $f \in A$, then $\overline{f} \in A$.

Then $A = C_0(X)$.

Proof: This is Corollary 5.8.3 in [10]. □

Another important theorem is the Hahn-Banach Theorem.

Definition 1.14 If X is a vector space, a sublinear functional is a function $q : X \to \mathbb{R}$ such that

1. $q(x + y) \leq q(x) + q(y)$ for all x, y in X;
2. $q(ax) = \alpha q(x)$ for x in X and $\alpha > 0$.

Theorem 1.15 (Hahn-Banach) *Let X be a vector space over $\mathbb{R}$ and let q be a sublinear functional on X. If M is a linear manifold in X and $f : M \to \mathbb{R}$ is a linear functional such that $f(x) \leq q(x)$ for all x in M, then there is a linear functional $F : X \to \mathbb{R}$ such that $F|_M = f$ and $F(x) \leq q(x)$ for all x in X.*

Proof: This is theorem 3.6.2 in [10]. □

Later on we will use some consequences of this theorem, namely 4.1 and 4.17

1.2 The Riesz Functional Calculus

We will discuss the Riesz Functional Calculus. We will only state the theorems, the proofs can be found in ([10], p199-p205). We assume the reader is familiar with the definition of a positively oriented system of curves in an open subset G of $\mathbb{C}$. If not, see ([10], p199-p205).

Let A be a Banach algebra over $\mathbb{C}$ with identity. Let G be an open subset of $\mathbb{C}$, γ a rectifiable curve in G and f a continuous function defined in a neighbourhood of $\{\gamma\}$ with values in A. Then we can define the integral $\int_\gamma f(z)dz$ as for a scalar-valued f in the following way. For every $k \in \mathbb{N}$, let $\{(t_k)_0, \cdots, (t_k)_{n_k}\}$ be a partition of $[0, 1]$ such that $|(t_k)_{j+1} - (t_k)_j| \to 0$ as $k \to \infty$ for all $0 \leq j < n_k$. Then we define

$$\int_\gamma f(z)dz = \lim_{k\to\infty} \sum_j (\gamma((t_k)_j) - \gamma((t_k)_{j-1}))f(\gamma((t_k)_j)),$$

Hence $\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))d\gamma(t) \in A$.

Proposition 1.16 *If G is an open subset of $\mathbb{C}$ and K is a compact subset of G, then there is a positively oriented system of curves $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ in $G \setminus K$ such that $K \subseteq \operatorname{ins} \Gamma$ and $\mathbb{C} \setminus G \subseteq \operatorname{out} \Gamma$. The curves $\gamma_1, \ldots, \gamma_m$ can be found such that they are infinitely differentiable.*

If $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is a positively oriented system of curves, define

$$\int_\Gamma f(z)dz = \sum_{j=1}^{m} \int_{\gamma_j} f(z)dz$$

whenever f is a continuous function in a neighborhood of $\{\Gamma\}$ with values in A.

Let $a \in A$. If $f : G \to \mathbb{C}$ is analytic and $\sigma(a) \subseteq G$, we define an element $f(a)$ in A by

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-a)^{-1}dz$$

where Γ is as in Proposition 1.16 with $K = \sigma(a)$. The following proposition tells us that $f(a)$ is well defined.

Proposition 1.17 *Let A be a Banach algebra with identity, let $a \in A$, and let G be an open subset of $\mathbb{C}$ such that $\sigma(a) \subseteq G$. If $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$ are two positively oriented collections of curves in G such that $\sigma(a) \subseteq \operatorname{ins} \Gamma \subseteq G$ and $\sigma(a) \subseteq \operatorname{ins} \Lambda \subseteq G$ and if $f : G \to \mathbb{C}$ is analytic, then*

$$\int_\Gamma f(z)(z-a)^{-1}dz = \int_\Lambda f(z)(z-a)^{-1}dz.$$

Let $\mathrm{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. We have the usual sum and product for functions. If $f, g \in \mathrm{Hol}(a)$ have domains $D(f)$ and $D(g)$, then fg and $f + g$ have domain $D(f) \cap D(g)$. From this it follows that $\mathrm{Hol}(a)$ is not an algebra. The domain of the zero function is $\mathbb{C}$, but if we have a function $f \in \mathrm{Hol}(a)$ with smaller domain, it follows that $0 = f - f$, has a smaller domain, thus a contradiction. We can however define an equivalence relation on $\mathrm{Hol}(a)$ such that the set of equivalence classes forms an algebra. We say that two functions are equivalent if they are equal on some open neighbourhood of $\sigma(a)$ and then $\mathrm{Hol}(a)/\sim$ is an algebra. We will denote this algebra also with $\mathrm{Hol}(a)$.

Theorem 1.18 (The Riesz Functional Calculus) *Let A be a Banach algebra with identity and let $a \in A$.*

1. *The map $f \mapsto f(a)$ from $\mathrm{Hol}(a)$ to A is an algebra homomorphism.*
2. *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence strictly larger than $r(a)$, then $f \in \mathrm{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*
3. *If $f(z) \equiv 1$, then $f(a) = 1$.*
4. *If $f(z) = z$ for all z, then $f(a) = a$.*
5. *If $f, f_1, f_2, \ldots$ are all analytic on G, $\sigma(a) \subseteq G$, and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.*

The following proposition tells us that the functional calculus is unique.

Proposition 1.19 *Let A be a Banach algebra with identity and let $a \in A$. Let $\tau : \mathrm{Hol}(a) \to A$ be a homomorphism such that (1) $\tau(1) = 1$, (2) $\tau(z) = a$, (3) if $\{f_n\}$ is a sequence of analytic functions on an open set G such that $\sigma(a) \subseteq G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\tau(f_n) \to \tau(f)$. Then $\tau(f) = f(a)$ for every f in $\mathrm{Hol}(a)$.*

Let $\tau : \mathrm{Hol}(a) \to A$ be the algebra homomorphism given by $\tau(f) = f(a)$. Then we have $f(a)g(a) = \tau(fg) = \tau(gf) = g(a)f(a)$. So for all $f, g \in \mathrm{Hol}(a)$, $f(a)$ and $g(a)$ commute. Still more can be said.

Proposition 1.20 *If $a, b \in A$, $ab = ba$, and $f \in \mathrm{Hol}(a)$, the $f(a)b = bf(a)$.*

Theorem 1.21 (The Spectral Mapping Theorem) *If $a \in A$ and $f \in Hol\ (a)$, then*

$$\sigma(f(a)) = f(\sigma(a)).$$

If λ is an isolated point in the spectrum of a, we can define the associated spectral projection:

Definition 1.22 If λ_0 is an isolated point in $\sigma(a)$, then there is an open subset G_1 of $\mathbb{C}$ with $G_1 \cap \sigma(a) = \lambda_0$. Let $G \subset \mathbb{C}$ be open with $\sigma(a) \subset G$ and let $G' = G \setminus \overline{G_1}$. Define $f : G_1 \cup G' \to \mathbb{C}$ to be 1 on G_1 and 0 on G', then $f \in \mathrm{Hol}(a)$. Now we define the *the spectral projection* associated to a and λ_0 as $p(a, \lambda_0) := f(a)$.

From the definition of the functional calculus we see that

$$p(a, \lambda_0) = \frac{1}{2\pi i} \int_\Gamma (\lambda - x)^{-1} d\lambda$$

with Γ any circle centered at λ_0 separating λ_0 from the rest of the spectrum.

Now we use the functional calculus to prove a lemma about the distance to a spectrum.

Lemma 1.23 *Let A be a Banach algebra. Suppose that $x \in A$ and that $\alpha \notin \sigma(x)$. Then we have*

$$\mathrm{d}(\alpha, \sigma(x)) = \frac{1}{r((\alpha - x)^{-1})}.$$

Proof: Let G be an open set containing $\sigma(x)$, but not α. Let $f(\lambda) = 1/(\alpha - \lambda)$, then f is holomorphic on G and the Spectral Mapping Theorem gives us

$$\sigma((\alpha - x)^{-1}) = \left\{ \frac{1}{\alpha - \lambda} : \lambda \in \sigma(x) \right\}.$$

So in particular,

$$\begin{aligned} r((\alpha - x)^{-1}) &= \sup \left\{ \frac{1}{|\alpha - \lambda|} : \lambda \in \sigma(x) \right\} \\ &= 1/\inf\{|\alpha - \lambda| : \lambda \in \sigma(x)\} = 1/\mathrm{d}(\alpha, \sigma(x)). \end{aligned}$$

□

1.3 The radical

Before we define the radical of a Banach algebra, we first say what ideals are.

Definition 1.24 If A is an algebra, a *left ideal* of A is a subalgebra M of A such that $ax \in M$ whenever $a \in A,\ x \in M$. A *right ideal* of A is a subalgebra M of A such that $xa \in M$ whenever $a \in A,\ x \in M$. A *(bilateral) ideal* is a subalgebra of A that is both a left ideal and a right ideal.

We call an ideal *proper* if it is neither 0 or A. If an ideal is contained in no larger proper ideal and is not equal to A, then it is a *maximal ideal.* If an ideal is not equal to A and the only ideals it contains are the zero ideal and itself, it is a *minimal ideal* . Every proper ideal of a unital algebra is contained in a maximal ideal (this is an application of Zorn's Lemma), but it does not necessarily have minimal ideals.

Proposition 1.25 *If A is a Banach algebra with identity, then*

1. *The closure of a proper left, right, or bilateral ideal is a proper left, right, or bilateral ideal,*
2. *A maximal left, right, or bilateral ideal is closed.*

Proof: This is Corollary 7.2.5. in [10] □

To define the radical, we need the following Theorem by N. Jacobson.

Theorem 1.26 (N. Jacobson) *Let A be an algebra with unit 1 and let $x, y \in A$, $\lambda \in \mathbb{C}$, with $\lambda \neq 0$. Then $\lambda - xy$ is invertible in A if and only if $\lambda - yx$ is invertible in A.*

Proof: Suppose that $\lambda - xy$ has an inverse $z \in A$: $(\lambda - xy)z = z(\lambda - xy) = 1$. Hence

$$\begin{aligned}(\lambda - yx)(yzx + 1) &= \lambda yzx + \lambda - y(xyz)x - yx \\ &= \lambda yzx + \lambda - y(\lambda z - 1)x - yx = \lambda,\end{aligned}$$

and

$$\begin{aligned}(yzx + 1)(\lambda - yx) &= \lambda yzx + \lambda - y(zxy)x - yx \\ &= \lambda yzx + \lambda - y(\lambda z - 1)x - yx = \lambda.\end{aligned}$$

Thus $\lambda - yx$ is invertible in A. □

Theorem 1.27 *Let A be a ring with unit 1. Then the following sets are identical:*

1. *The intersection of all maximal left ideals A.*
2. *The intersection of all maximal right ideals of A.*
3. $\{x \in A : 1 - zx$ *is invertible for all* $z \in A\}$.
4. $\{x \in A : 1 - xz$ *is invertible for all* $z \in A\}$.

Proof: By the preceding lemma, the sets in (3) and (4) are identical. We will prove that the sets in (1) and (3) are identical. With a similar argument we can prove that the sets in (2) and (4) are identical.

Let x be in the intersection of all maximal left ideals. Suppose there is a $z \in A$ such that $1 - zx$ is not invertible. We show that there exists a $z' \in A$ such that $1 - z'x$ is not left invertible. Suppose $1 - z'x$ is left invertible for all $z' \in A$. Then this is in particular true for z, so there exists a $y \in A$ such that $(1 - y)(1 - zx) = 1$, from which it follows that $y = (-z + yz)x$. So by our assumption $1 - y = 1 - (-z + yz)x$ is left invertible. Since $(1 - y)(1 - zx) = 1$ it is also right invertible, thus invertible. But this means that $1 - zx = (1 - y)^{-1}$ is invertible, which is a contradiction. Let $z' \in A$ be such that $1 - z'x$ is not left invertible. Then $1 - z'x$ is contained in a maximal left ideal M. But $x \in M$, so $1 \in M$ which is a contradiction.

Conversely, let $x \in A$ such that $1 - zx$ is invertible for all $z \in A$. Suppose x is not in the intersection of all maximal left ideals. Then there is a maximal left ideal M such that $x \notin M$. So $M + Ax = A$, thus $1 - zx \in M$ for some z in A. But this is a contradiction since $1 - zx$ is invertible for all z in A. □

Definition 1.28 If A is a Banach algebra with identity then the set having properties (1)-(4) is called the radical, $\mathrm{Rad}(A)$, of A. If $\mathrm{Rad}(A) = 0$, we say that A is semi-simple.

It is clear that $\mathrm{Rad}(A)$ is a two-sided ideal of A.

Definition 1.29 If $a \in A$, then a is called quasinilpotent if $\sigma(a) = \{0\}$. The set of quasinilpotent elements in A will be denoted by $QN(A)$.

Theorem 1.30 $\mathrm{Rad}(A) \subset \mathrm{QN}(A)$.

Proof: From Theorem 1.27 we see that if $a \in \mathrm{Rad}(A)$, then $1 - az$ is invertible for all $z \in A$. So $\lambda - a$ is invertible for $\lambda \neq 0$ in $\mathbb{C}$, which implies that $r(a) = 0$. Thus $a \in \mathrm{QN}(A)$. □

Theorem 1.31 $\mathrm{Rad}(A) = \{a \in A : aA \subset \mathrm{QN}(A)\} = \{a \in A : Aa \subset \mathrm{QN}(A)\}$.

Proof: For the first equality, let $a \in \{a \in A : aA \subset \mathrm{QN}(A)\}$. Then $\sigma(az) = 0$ for all $z \in A$. So $1 - az$ is invertible for all $z \in A$, which implies that $a \in \mathrm{Rad}(A)$ by Theorem 1.27. The other inclusion follows from the previous theorem. The second equality follows in the same way. □

1.4 Inessential ideals

Let A be a Banach algebra and X a complex vector space of dimension greater than or equal to one. A representation of A on X is a non-zero algebra homomorphism π from A into the algebra $L(X)$ of linear operators on X. Let $Y \subset X$, then Y is invariant under π if for all $x \in A$ we have $\pi(x)Y \subset Y$. A representation is irreducible if the only linear subspaces of X invariant under π are $\{0\}$ and X. A representation is called bounded if X is a Banach space and

if $\pi(x)$ is a bounded linear operator for all $x \in A$. Moreover it is continuous if there exists a constant $C > 0$ such that $\|\pi(x)\| \leq C\|x\|$ for all $x \in A$, so continuous implies bounded. It is easy to see that the kernel of a continuous representation is closed.
Let I be a not necessarily closed two-sided ideal of a Banach algebra A. We say that I is *inessential* if the spectrum of every element in the ideal is either finite or a sequence converging to zero. Given a two-sided ideal I of A we denote by $\mathrm{kh}(I)$ the intersection of all kernels of continuous irreducible representations π of A such that $I \subset \ker(\pi)$. Since $\ker(\pi)$ is closed, we see that $I \subset \overline{I} \subset \mathrm{kh}(I)$. It can be shown, ([5],theorem 4.2.1), that the radical of A is the intersection of the kernels of all continuous irreducible representations of A. From this one can quite easily deduce that $\mathrm{kh}(I)$ is the inverse image of $\mathrm{Rad}(A/\overline{I})$. We call an element a in A inessential relative to I if $a \in \mathrm{kh}(I)$, i.e. if $\overline{a} \in \mathrm{Rad}(A/\overline{I})$.

Theorem 1.32 *Let I be a two-sided ideal of A and let $x \in \mathrm{kh}(I)$. Suppose that $\alpha \neq 0$ is isolated in $\sigma(x)$. Then the spectral projection $p(x, \alpha)$ is in I.*

Proof: Let Γ be a circle centered at α, separating α from 0 and from the rest of the spectrum. For $\lambda \in \Gamma$ we have

$$(\lambda - x)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda} x(\lambda - x)^{-1}.$$

So

$$p(x, \alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{d\lambda}{\lambda} + \frac{x}{2\pi i} \int_\Gamma \frac{1}{\lambda} (\lambda - x)^{-1} d\lambda.$$

The first term is zero and since $x \in \mathrm{kh}(I)$, the second term is in $\mathrm{kh}(I)$. So $p \in \mathrm{kh}(I)$ and we have that $\overline{p} \in \mathrm{Rad}(A/\overline{I})$. Therefore $r(\overline{p}) = 0$ and in particular $(1 - \overline{p})$ is invertible. Since p is a projection, $\overline{p}$ is a projection and we have $\overline{p}(1 - \overline{p}) = 0$, so $\overline{p} = 0$. Hence $p \in \overline{I}$. Moreover $p\overline{I}p$ is a closed subalgebra of A, hence a Banach algebra with identity p. In this subalgebra pIp is a dense two-sided ideal. The closure of a proper two-sided ideal is again proper by Proposition 1.25, so pIp is not proper, thus $pIp = p\overline{I}p$. Therefore $p = p^3 \in p\overline{I}p = pIp \subset I$. □

Corollary 1.33 *Let I be a two-sided ideal of A. Then I and $\mathrm{kh}(I)$ have the same set of idempotents.*

Proof: The corollary follows from the last part of the proof of Theorem 1.32. □

If A has minimal left ideals (resp. right ideals), then its *socle*, $\mathrm{soc}(A)$, is defined as the sum of the minimal left ideals. It is also equal to the sum of the minimal right ideals ([5], p110) and therefore is a two-sided ideal. If A is semisimple, then $\mathrm{soc}(A)$ exists and is inessential (see [6]). For more information on the socle we refer to ([2], p78-p87)

An element a in A is called *Riesz* relative to a closed ideal J if the spectrum of the element $\overline{a}$ in the quotient algebra A/J consists of zero. We denote the set of Riesz elements in A relative to J by $R(A, J)$ or $R(J)$ if it is clear what is meant. From Theorem 1.30 it follows that $\mathrm{kh}(J) \subset R(A, J)$. We call an isolated point $\lambda \in \sigma(a)$ a *Riesz point of* $\sigma(a)$ relative to an ideal J (not necessarily closed) if the corresponding spectral projection $p(a, \lambda)$ belongs to J.

Let J be a two-sided not necessarily closed inessential ideal of A. Following [4] we define for an element $a \in A$ the set $D(a, A, J)$ as follows:

Definition 1.34 Let A be a Banach algebra, J an inessential ideal in A and $a \in A$, then

$$D(a, A, J) = \sigma(a) \setminus \{\lambda \in \sigma(a) : \lambda \text{ is a Riesz point of } \sigma(a) \text{ relative to } J\} .$$

If it is clear what is meant, we shall just write $D(a)$ and say that λ is a Riesz point of $\sigma(a)$. It is easy to verify that $D(a, A, J)$ is compact and that $\sigma(a) \setminus D(a)$ is discrete and hence countable.

Definition 1.35 If A is a set and $f : A \to \mathbb{C}$, define

$$\|f\|_A := \sup\{|f(z)| : z \in A\}.$$

If K is a compact subset of $\mathbb{C}$, define the *polynomially convex hull* of K to be the set $K^\wedge$, given by

$$K^\wedge := \{z \in \mathbb{C} : |p(z)| \leq \|p\|_K \text{ for every polynomial } p\}.$$

The set K is *polynomially convex* if $K = K^\wedge$.

If K is a compact set, then $\mathbb{C} \setminus K$ has a countable number of components, only one of which is unbounded. The bounded components are called the holes of K. The *connected hull* ηK of K has as its complement the unbounded component of $\mathbb{C}\setminus K$. Thus ηK is the union of K and its holes and thus $\eta K = K^\wedge$ ([10], 7.5.3).

From now on, if we speak of an ideal we mean a two-sided ideal.

Theorem 1.36 (Perturbation by Inessential Elements) *Let I be an inessential ideal of a Banach algebra A. For $x \in A$ and $y \in I$ we have the following properties:*

1. *if G is a connected component of $\mathbb{C} \setminus D(x)$ intersecting $\mathbb{C} \setminus \sigma(x + y)$ then it is a connected component of $\mathbb{C} \setminus D(x + y)$,*
2. *the unbounded connected components of $\mathbb{C}\setminus D(x)$ and $\mathbb{C}\setminus D(x+y)$ coincide, in particular $D(x)$ and $D(x + y)$ have the same external boundaries,*
3. *if $\overline{x}$ denotes the class of x in $A/\overline{I}$ then we have $\sigma(\overline{x}) \subset D(x)$ and $D(x)^\wedge = \sigma(\overline{x})^\wedge$.*

Proof: This is Theorem 2.4 in [4]. □

When $B \subset A$ is a subalgebra of A and I an ideal in both A and B, it is not clear what is meant with the closure of I. Therefore we introduce the following notation: the closure of I in B is denoted by I_B and the closure of I in A by I_A.

Theorem 1.37 *Let A en B be Banach algebras such that $B \subset A$ is a subalgebra of A and such that $1 \in B$. Suppose that I is an inessential ideal both in A and in B such that $I_B \subset I_A$. For arbitrary $a \in B$ consider the following statements.*

(a) $\sigma(\overline{a}, A/I_A) = \sigma(\overline{a}, B/I_B)$.

(b) $\sigma(a+b,A)=\sigma(a+b,B)$ *for every* $b\in I_B$.

(c) $\sigma(a,A)=\sigma(a,B)$.

(d) $D(a,A,I)=D(a,B,I)$.

Then the following implications are valid:

$$(a)\Rightarrow(b)\Rightarrow(c)\Leftrightarrow(d)$$

The implications (b) ⇒ (a) and (c) ⇒ (b) are false.

Proof: This is Theorem 5.4 in [11]. □

Corollary 1.38 *Let I be an inessential ideal of a Banach algebra A. Let $a\in A$ and suppose that $r(\overline{a},A/\overline{I})=0$. Then $\sigma(a)$ is either finite or a sequence converging to zero and for every non-zero value of $\sigma(a)$ the associated spectral projection is in I.*

Proof: If $r(\overline{a})=0$, by Theorem 1.36.3 we have $D(a)^\wedge=\{0\}$, so $D(a)=\{0\}$. □

Corollary 1.39 *Let I be a two-sided inessential ideal of a Banach algebra A. Then $\mathrm{kh}(I)$ is inessential, so in particular $\overline{I}$ is inessential.*

Proof: If $x\in\mathrm{kh}(I)$ then $\overline{x}\in\mathrm{Rad}(A/\overline{I})$, so $r(\overline{x})=0$. We apply the previous corollary. □

Theorem 1.40 (Ruston characterisation) *Let A be a Banach algebra, $a\in A$ and I a closed inessential ideal of A. Then $a\in R(A,I)$ if and only if $\sigma(a)$ is finite or a sequence converging to zero and all non-zero elements of $\sigma(a)$ are Riesz points relative to I.*

Proof: If $a\in R(A,I)$ then $\sigma(\overline{a})=0$ and the required properties follow from Corollary 1.38.

Conversely, let $\sigma(a)$ be finite or a sequence converging to zero, and suppose that for every $0\neq\alpha\in\sigma(a)$ the spectral projection $p(a,\alpha)$ lies in I. We have to prove that $\sigma(\overline{a})=\{0\}$. Let $0\neq\alpha\in\sigma(a)$, then $p:=p(a,\alpha)\in I$, so that $\overline{p}=\overline{0}$. Let G_1,G_2 be disjoint open subsets of $\mathbb{C}$ such that $\alpha\in G_1$ and $\sigma(a)\setminus\{\alpha\}\subset G_2$. If f is the characteristic function of G_1, then $f\in\mathrm{Hol}(a)$ and $f(a)=p$. Because $\sigma(\overline{a})\subset\sigma(a)$, $f\in\mathrm{Hol}(\overline{a})$ and $f(\overline{a})=\overline{f(a)}=\overline{0}$. The spectral mapping theorem yields $f(\sigma(\overline{a}))=\sigma(f(\overline{a}))=\{0\}$. This holds for every $0\neq\alpha\in\sigma(a)$, so $\sigma(\overline{a})=\{0\}$ and therefore $a\in R(A,I)$. □

1.5 Ordered Banach algebras

In this section, following [26], we will define an algebra cone C of a real or complex Banach algebra A and show that C induces on A an ordering which is compatible with the algebraic structure of A. The Banach algebra A is then called an ordered Banach algebra (OBA). We also define certain additional properties of C.

Definition 1.41 Let A be a real or complex Banach algebra with unit 1. We call a nonempty subset C of A a *cone* if it satisfies the following:

1. $C + C \subset C$,
2. $\lambda C \subset C$ for all $\lambda \geq 0$.

If in addition C satisfies $C \cap -C = \{0\}$, then C is called a *proper* cone.

Any cone C on A induces a relation '$\leq$' on A, called an ordering, in the following way:

$$a \leq b \quad \text{if and only if} \quad b - a \in C, \quad (a, b \in A).$$

It can be shown that for every $a, b \in A$ this ordering satisfies

1. $a \leq a \quad$ ($\leq$ is reflexive),
2. if $a \leq b$ and $b \leq c$, then $a \leq c \quad$ ($\leq$ is transitive).

The ordering does not have to be antisymmetric.

Proposition 1.42 *The cone C is proper if and only if the ordering is antisymmetric, i.e. $a \leq b$ and $b \leq a$ implies that $a = b$.*

Proof: Let C be a proper cone, $a \leq b$ and $b \leq a$. Then $a - b \in C$ and $b - a = -(a - b) \in C$, so $a - b \in C \cap -C = \{0\}$ and we have $a = b$.

Conversely, let the ordering be antisymmetric and suppose the cone C is not proper. Then there exists an $x \in C$ with $x \neq 0$ such that there is an $a \in C$ with $x = -a$. Now we have $x - a = 2x \in C$ and $a - x = 2a \in C$. So $x \leq a$ and $a \leq x$ and the antisymmetric property gives us $x = a$, which is a contradiction. $\square$

So the ordering induced by C is a partial ordering if and only if C is proper.

Considering the ordering that C induces, we find that $C = \{a \in A : a \geq 0\}$, and therefore we call the elements of C positive.

Definition 1.43 A cone C of a Banach algebra A is called an *algebra cone* if C satisfies the following conditions:

1. $C \cdot C \subset C$,
2. $1 \in C$.

Definition 1.44 A real or complex Banach algebra A with unit 1 is called an *ordered Banach algebra* (OBA) if A is ordered by a relation '$\leq$' in such a manner that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$ we have:

1'. $a, b \geq 0 \Rightarrow a + b \geq 0$,

2'. $a \geq 0,\ \lambda \geq 0 \Rightarrow \lambda a \geq 0$,

3'. $a, b \geq 0 \Rightarrow ab \geq 0$,

4'. $1 \geq 0$.

Therefore, if A is ordered by an algebra cone C, then A, or more specifically (A, C) is an OBA. Conversely, if A is an OBA the set $C = \{a \in A : a \geq 0\}$ is an algebra cone that induces the ordering on A.

Definition 1.45 An algebra cone C is called *normal* if there exists a constant $\beta \geq 1$ such that for $a, b \in A$ we have

$$0 \leq a \leq b \quad \Rightarrow \quad \|a\| \leq \beta \|b\|$$

An alternative definition of normality is,

Definition 1.46 An algebra cone C is called *α-normal* if there exists a constant $\alpha \geq 1$ such that for $a, b, c \in A$ we have

$$a \leq b \leq c \quad \Rightarrow \quad \|b\| \leq \alpha(\max \|a\|, \|c\|).$$

If the normality constant α is equal to 1 we say that the C is 1-normal.

It is not hard to prove that the two definitions are equivalent, but the constants β and α from the definitions need not be the same. If C is normal with constant α, C does not have to be α-normal.

Proposition 1.47 *If C is a normal algebra cone, then it is a proper algebra cone.*

Proof: Let C be a normal algebra cone. Let $x \in C$ be such that there exists an $a \in C$ with $x = -a$. Then for all scalars $k > 0$ we have $a - ka = a + k(-a) = a + kx \in C$, so $ka \leq a$. Because C is normal there exists a constant $\alpha > 0$ such that for all $k > 0$ we have $k\|a\| = \|ka\| \leq \alpha \|a\|$, so $\|a\| = 0$. This means that $a = 0$ and therefore $C \cap -C = \{0\}$. □

If C has the property that if $a \in C$ and a is invertible, then $a^{-1} \in C$, then C is said to be inverse-closed. The following lemma is immediate.

Lemma 1.48 *Let (A, C) be an OBA, and let $x, y \in A$ be such that $xy \leq yx$.*

1. *If x is invertible and $x^{-1} \in C$, then $yx^{-1} \leq x^{-1}y$.*
2. *If y is invertible with $y^{-1} \in C$, then $y^{-1}x \leq xy^{-1}$.*

The following lemma follows with induction.

Lemma 1.49 *Let (A, C) be an OBA, and let $x, y \in C$. If $yx \leq xy$, then*

$$(x + y)^n \leq \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

for every $n \in \mathbb{N} \cup \{0\}$.

Proof: The statement clearly is true for $n = 0$. Now let $m > 0$ and suppose the statement is true for all $n < m$. We have that $yx \leq xy$ implies $yx^{m-k-1}y^k \leq$

$x^{m-k-1}y^{k+1}$, and it follows that

$$\begin{aligned}
(x+y)^m &\leq (x+y)\sum_{k=0}^{m-1}\binom{m-1}{k}x^{m-k-1}y^k \\
&\leq \sum_{k=0}^{m-1}\binom{m-1}{k}x^{m-k}y^k + \sum_{k=0}^{m-1}\binom{m-1}{k}x^{m-k-1}y^{k+1} \\
&= x^m + \sum_{k=1}^{m-1}\left(\binom{m-1}{k} + \binom{m-1}{k-1}\right)x^{m-k}y^k + y^m \\
&= \sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k.
\end{aligned}$$

□

Let A and B be Banach algebras such that $1 \in B \subset A$, then we have a few easy to prove facts.

(i) If C is an algebra cone in A, then $C \cap B$ is an algebra cone in B and if C is proper, then $C \cap B$ is proper.

(ii) In the case where B has a finer norm then A, (i.e $\|b\|_A \leq \|b\|_B$ for all $b \in B$) if C is closed in A, then $C \cap B$ is closed in B.

(iii) If B is a closed subalgebra of A (containing the unit of A), then the normality of C in A implies normality of $C \cap B$ in B.

(iv) If $T : A \to B$ is a homomorphism and if C is an algebra cone of A, then $TC = \{Tc : c \in C\}$ is an algebra cone in B. In particular, if F is a closed ideal in the OBA (A, C) and if $\pi : A \to A/F$ is the canonical homomorphism, then πC is an algebra cone of A/F. We cannot deduce normality or closedness of πC from the corresponding properties of C.

Now we give some examples of OBA's.

Example 1.50 *Let $A = \mathbb{C}$ be the Banach algebra with standard norm and $C = \mathbb{R}^+$. Then (A, C) is an OBA and C is normal.*

Proof: Trivial. □

Example 1.51 *Let $\mathbb{C}^2$ be equipped with $\|\cdot\|_\infty$ and let A be the set of upper triangular 2×2 complex matrices with the operator norm for bounded operators. Let C the subset of A of matrices with only nonnegative entries. Then (A, C) is an OBA and C is normal.*

Proof: It follows from simple calculations that (A, C) is an OBA. From the definition of the operator norm for bounded operators we have for $M \in A$,

$$\begin{aligned}
\|M\| &= \max\{\|Mx\|_\infty : x \in \mathbb{C}^2 \text{ with } \|x\|_\infty \leq 1\} \\
&= \max\{|m_{11}| + |m_{12}|, |m_{21}| + |m_{22}|\}.
\end{aligned}$$

Let $M, N \in A$ with $0 \leq M \leq N$, then $m_{ij} \leq n_{ij}$ for all $i, j \in \{1, 2\}$. Thus we see from the definition of the norm that $\|M\| \leq \|N\|$. $\square$

Now an example of an infinite-dimensional and semisimple OBA, l^∞, consisting of all bounded sequences of complex numbers.

Example 1.52 *Let $A = l^\infty$ with multiplication defined coordinatewise and $C = \{(c_1, c_2, \cdots) \in l^\infty : c_i \geq 0$ for all $i \in \mathbb{N}\}$. Then (A, C) is an OBA, A is semisimple and C is normal.*

Proof: From the coordinatewise multiplication it follows easily that A is a Banach algebra, with unit $(1, 1, \cdots)$. Direct calculation shows that C is an algebra cone. Now we show that C is normal. Suppose that $(0, 0, \cdots) \leq (x_1, x_2, \cdots) \leq (y_1, y_2, \cdots)$ in A. By definition of C this means that $0 \leq x_k \leq y_k$ for all $k \in \mathbb{N}$. Hence $\|(x_1, x_2, \cdots)\| \leq \|(y_1, y_2, \cdots)\|$, thus C is normal.

We have $\sigma((x_1, x_2, \cdots)) = \overline{\{x_1, x_2, \cdots\}}$ for $(x_1, x_2, \cdots) \in l^\infty$, so $\mathrm{QN}(l^\infty) = \{0\}$. It follows form Theorem 1.30 that $\mathrm{Rad}(l^\infty) = \{0\}$, i.e. l^∞ is semisimple. $\square$

Now we look at the set consisting of all bounded sequences of upper triangular 2×2 complex matrices to get an example of an infinite-dimensional and not semisimple OBA.

Example 1.53 *Let A be the set of upper triangular 2×2 matrices, $l^\infty(A)$ the set*

$$\{x = (x_1, x_2, \cdots) : x_i \in A \text{ and } \|x_i\|_A \leq K_x \text{ for some } K_x \in \mathbb{R}, \text{ for all } i \in \mathbb{N}\},$$

and C the set

$$\{(c_1, c_2, \cdots) \in l^\infty(A) : c_i \text{ has only nonnegative entries for all } i \in \mathbb{N}\}.$$

Then $(l^\infty(A), C)$ is an OBA, C is closed and normal and $l^\infty(A)$ is not semisimple.

Proof: By defining addition, scalar multiplication and multiplication coordinatewise and the norm to be $\|(x_1, x_2, \cdots)\| = \sup_{j \in \mathbb{N}} \|x_j\|_A$ it is not hard to show that $l^\infty(A)$ is an Banach algebra with unit $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \cdots\right)$. Direct calculation also show that C is an algebra cone of $l^\infty(A)$. Now we will prove normality. Suppose $0 \leq x \leq y$, where

$$0 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \cdots\right), \; x = \left(\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{14} \end{pmatrix}, \begin{pmatrix} x_{21} & x_{22} \\ 0 & x_{24} \end{pmatrix}, \cdots\right)$$

and

$$y = \left(\begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{14} \end{pmatrix}, \begin{pmatrix} y_{21} & y_{22} \\ 0 & y_{24} \end{pmatrix}, \cdots\right)$$

From the definition of C we see that $0 \leq x_{jk} \leq y_{jk}$ for all $j \in \mathbb{N}$ and $k = 1, 2, 4$. Therefore $\max\{|x_{j1}| + |x_{j2}|, |x_{j4}|\} \leq \max\{|y_{j1}| + |y_{j2}|, |y_{j4}|\}$, i.e.

$$\left\|\begin{pmatrix} x_{j1} & x_{j2} \\ 0 & x_{j4} \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} y_{j1} & y_{j2} \\ 0 & y_{j4} \end{pmatrix}\right\|,$$

for all $j \in \mathbb{N}$. It follows that

$$\sup_{j\in\mathbb{N}} \left\| \begin{pmatrix} x_{j1} & x_{j2} \\ 0 & x_{j4} \end{pmatrix} \right\| \leq \sup_{j\in\mathbb{N}} \left\| \begin{pmatrix} y_{j1} & y_{j2} \\ 0 & y_{j4} \end{pmatrix} \right\|,$$

i.e. $\|x\| \leq \|y\|$. Thus C is normal. The closedness of C follows easily from the definition of C.

Since $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \cdots \right)$ is an element of the radical, $l^\infty(A)$ is not semisimple. $\square$

Chapter 2

Spectral properties in OBA's

In this chapter we will establish properties of the spectral radius in an OBA. We will follow [26] and [19].

Definition 2.1 Let (A, C) be an OBA. If $0 \leq a \leq b$ relative to C implies $r(a) \leq r(b)$, then we say that the spectral radius (function) is monotone w.r.t. the algebra cone C.

Theorem 2.2 *Let (A, C) be an OBA with a normal algebra cone C. Then the spectral radius is monotone w.r.t. C.*

Proof: Let $0 \leq a \leq b$, then we see with induction that $0 \leq a^n \leq b^n$. Let α be the normality constant, then $\|a^n\| \leq \alpha\|b^n\|$ for all $n \in \mathbb{N}$, so $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \leq \lim_{n\to\infty}(\alpha\|b^n\|)^{1/n} = \lim_{n\to\infty} \alpha^{1/n} \cdot \lim_{n\to\infty} \|b^n\|^{1/n} = r(b)$. $\square$

Theorem 2.3 *Let (A, C) be an OBA with algebra cone C such that the spectral radius is monotone. Let $a, b \in A$ be such that $0 \leq a \leq b$ relative to C. Then*

1. *if b is quasinilpotent then a is quasinilpotent,*
2. *if b is in the radical of A then a is quasinilpotent,*
3. *if b is in the radical of A and a in the center of A then a is in the radical of A.*

Proof:

1. $r(b) = 0$, so from Theorem 2.2 we have $0 \leq r(a) \leq 0$ wich gives $\sigma(a) = 0$.
2. From 1. and Theorem 1.30 we have $b \in \text{Rad } A \Rightarrow b \in QN(A) \Rightarrow a$ is quasinilpotent.
3. By 2. $r(a) = 0$. Let x be any element of A. Then, since a commutes with x, $r(ax) \leq r(a)r(x) = 0$, so $aA \subset QN(A)$. This implies that a is in the radical of A, by Theorem 1.31.

□

The converse of theorem 2.2 is in general not true. Also if the algebra cone is not normal, the spectral radius may not be monotone. Examples of both cases can be found in [26].

Proposition 2.4 *Let (A, C) be an OBA with normal algebra cone C and $a, b \in C$. If $ab \leq ba$ then $r(ba) \leq r(b)r(a)$, $r(ab) \leq r(a)r(b)$ and $r(a+b) \leq r(a)+r(b)$.*

Proof: If $a, b \in C$ with $ab \leq ba$, then $0 \leq (ba)^k \leq b^k a^k$ ($k \in \mathbb{N}$). The normality of C implies that $\|(ba)^k\| \leq \alpha \|b^k\| \|a^k\|$. As in the proof of 2.2 it follows that $r(ba) \leq r(b)r(a)$.

The second inequality follows in the same way as in the first part, from the observation that $(ab)^k \leq (ba)^k \leq b^k a^k$ for every $k \in \mathbb{N}$.

The last inequality will be proved in Theorem 6.14. □

Now we will discuss some results on the connection between the monotonicity of the spectral radius relative to algebra cones of different Banach algebras.

Proposition 2.5 *Let (A, C) be an OBA and B a Banach algebra with $1 \in B \subset A$ and such that the spectral radius function in the OBA $(B, C \cap B)$ is monotone. If $a, b \in B$ such that $0 \leq a \leq b$ relative to C and $r(b, B) = r(b, A)$ then $r(a, A) \leq r(b, A)$.*

Proof: Let $a, b \in B$ with $0 \leq a \leq b$ relative to C. Since the spectral radius in $(B, C \cap B)$ is monotone, $r(a, B) \leq r(b, B)$. Because B is a subalgebra of A we have $\sigma(a, A) \subset \sigma(a, B)$ and therefore $r(a, A) \leq r(a, B)$. We assumed $r(b, A) = r(b, B)$ and we get $r(a, A) \leq r(a, B) \leq r(b, B) = r(b, A)$. □

If we restrict ourselves to inessential ideals, we can prove a quite similar result in quotient algebras.

Theorem 2.6 *Let (A, C) be an OBA and B a Banach algebra such that $1 \in B \subset A$. Suppose that I is an inessential ideal of both A and B such that $I_B \subset I_A$, and such that the spectral radius function in the OBA $(B/I_B, \pi(C \cap B))$ is monotone. If $a, b \in B$ with $0 \leq a \leq b$ relative to C and $\sigma(b, B) = \sigma(b, A)$, then $r(\overline{a}, B/I_B) \leq r(\overline{b}, B/I_B)$ and $r(\overline{a}, A/I_A) \leq r(\overline{b}, A/I_A)$.*

Proof: Let $a, b \in B$ with $0 \leq a \leq b$ relative to C. Then $\overline{0} \leq \overline{a} \leq \overline{b}$ w.r.t. the algebra cone $\pi(C \cap B)$ of B/I_B. Because the spectral radius in $(B/I_B, \pi(C \cap B))$ is monotone, $r(\overline{a}, B/I_B) \leq r(\overline{b}, B/I_B)$. Let $\overline{a} \in B/I_B$ be invertible, then there exists a $\overline{c} \in B/I_B$ such that $\overline{a} \cdot \overline{c} = \overline{1}$. Since $I_B \subset I_A$, we have $\overline{a} \cdot \overline{c} = \overline{1}$ in A/I_A as well, so $\sigma(\overline{a}, A/I_A) \subset \sigma(\overline{a}, B/I_B)$ and therefore $r(\overline{a}, A/I_A) \leq r(\overline{a}, B/I_B)$. Theorem 1.37 and the assumption $\sigma(b, B) = \sigma(b, A)$ imply that $D(b, B, I) = D(b, A, I)$. The ideals I_A and I_B are inessential by Corollary 1.39, so Theorem 1.36.3 now tells us that $\sigma(\overline{b}, B/I_B)^\wedge = D(b, B, I)^\wedge = D(b, A, I)^\wedge = \sigma(\overline{b}, A/I_A)^\wedge$. So, $r(\overline{b}, B/I_B) = r(\overline{b}, A/I_A)$. Combining the results, it follows that $r(\overline{a}, A/I_A) \leq r(\overline{b}, A/I_A)$. □

Theorem 2.7 *Let (A, C) be an OBA with a closed algebra cone C such that the spectral radius function is monotone. If $a \in C$ then $r(a) \in \sigma(a)$.*

Proof: Let $a \geq 0$ and assume $r(a) = 1$. Suppose $1 \notin \sigma(a)$. Choose $0 < \alpha < 1$ such that $\sigma(a) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq \alpha\}$. Let t be a positive real number and let $f(z) := e^{tz}$. By the spectral mapping theorem $\sigma(e^{ta}) = e^{t\sigma(a)} \subset \{\lambda \in \mathbb{C} : |\lambda| \leq e^{t\alpha}\}$ and so $r(e^{ta}) \leq e^{t\alpha}$ for all $t \geq 0$. Since $a \in C$, $t \geq 0$ and C is a closed algebra cone, we have $e^{ta} = 1 + ta + ((ta)^2)/(2!) + \cdots \in C$ so that $0 \leq (t^n)/(n!)a^n \leq e^{ta}$, for all $n \in \mathbb{N}$ and $t \geq 0$. By the monotonicity of the spectral radius and $r(a) = 1$ we get, $0 \leq r((t^n)/(n!)a^n) = (t^n)/(n!) \leq e^{t\alpha}$. Substituting $t = n/\alpha$ in this inequality yields a contradiction to Stirling's formula. Hence $1 \in \sigma(a)$. $\square$

This theorem is a stronger version of the following well known theorem:

Theorem 2.8 *Let (A, C) be an OBA with a closed normal algebra cone C and $a \in C$. Then $r(a) \in \sigma(a)$.*

Proof: Because C is normal, the spectral radius is monotone by Theorem 2.2 and the result follows from Theorem 2.7. $\square$

Theorem 2.9 *Let (A, C) be an OBA with a closed cone C and let F be a closed ideal of A such that the spectral radius function in $(A/F, \pi C)$ is monotone. If $a \in C$ then $r(\overline{a}, A/F) \in \sigma(\overline{a}, A/F)$.*

Proof: Since we cannot deduce from the closedness of C that πC is closed, Theorem 2.9 does not just follow from Theorem 2.7, but the proof is almost the same as that of theorem 2.7. There is just one difference to get to the conclusion that $e^{t\overline{a}} \in \pi C$. Let $a \in C$, then we have $\pi(e^{ta}) = 1 + t\overline{a} + ((t\overline{a})^2)/(2!) + \cdots = e^{t\overline{a}}$ and $e^{ta} \in C$ because C is closed, so $e^{t\overline{a}} \in \pi C$. $\square$

Theorem 2.10 *Let (A, C) be an OBA and B a Banach algebra with $1 \in B \subset A$ such that $C \cap B$ is closed in B. Suppose that I is an inessential ideal of both A and B such that $I_B \subset I_A$ and suppose the spectral radius function in the OBA $(B/I_B, \pi(C \cap B))$ is monotone. If $a \in C \cap B$ is such that $\sigma(a, A) = \sigma(a, B)$, then $r(\overline{a}, B/I_B) \in \sigma(\overline{a}, B/I_B)$ and $r(\overline{a}, A/I_A) \in \sigma(\overline{a}, A/I_A)$.*

Proof: It follows from Theorem 2.9 that $r(\overline{a}, B/I_B) \in \sigma(\overline{a}, B/I_B)$. Theorem 1.37, Theorem 1.39 and the assumption $\sigma(a, B) = \sigma(a, A)$ imply that $D(a, B, I) = D(a, A, I)$. So $D(a, B, I)^\wedge = D(a, A, I)^\wedge$ and by Theorem 1.36.3 we have $\sigma(\overline{a}, A/I_A)^\wedge = \sigma(\overline{a}, B/I_B)^\wedge$. Hence $r(\overline{a}, A/I_A) = r(\overline{a}, B/I_B)$. Combining the results it follows that $r(\overline{a}, A/I_A) \in \sigma(\overline{a}, A/I_A)^\wedge$. Consider the polynomial $x + r(\overline{a}, A/I_A)$, then we have that $|2r(\overline{a}, A/I_A)| \leq \|x + r(\overline{a}, A/I_A)\|_{\sigma(\overline{a}, A/I_A)}$, and we conclude that $r(\overline{a}, A/I_A) \in \sigma(\overline{a}, A/I_A)$. $\square$

Theorem 2.11 *Let (A, C) be an ordered Banach algebra with C closed, normal and inverse-closed. If $a \in C$, then $\delta(a) \in \sigma(a)$.*

Proof: If a is not invertible then $\delta(a) = 0 \in \sigma(a)$. Suppose a is invertible. Since $a \in C$ and C is inverse-closed we have $a^{-1} \in C$. Also, because C is normal and closed, it follows from Theorem 2.8 that $r(a^{-1}) \in \sigma(a^{-1})$. So using the spectral mapping theorem we see that $r(a^{-1}) = 1/\lambda_0$, for some $\lambda_0 \in \sigma(a)$.

We have that $r(a^{-1}) = 1/\delta(a)$ by Lemma 1.23, which implies $\delta(a) = \lambda_0 \in \sigma(a)$.
$\square$

Lemma 2.12 *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Let $a \in C$.*

1. *If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(\overline{a}) < r(a)$.*
2. *If, in addition, the spectral radius in (A, C) is also monotone, then $r(a)$ is a Riesz point of $\sigma(a)$ if and only if $r(\overline{a}) < r(a)$.*

Proof: (1) If $r(\overline{a}) = r(a)$, then, by Theorem 2.9, $r(a) \in \sigma(\overline{a})$. Therefore by Theorem 1.36.3 $r(a) \in D(a)$, so that $r(a)$ is not a Riesz point of $\sigma(a)$.

(2) Conversely, if $r(\overline{a}) \leq r(a)$ and $r(a)$ is not a Riesz point of $\sigma(a)$, then by Theorem 2.7 we have $r(a) \in D(a)$, so by Theorem 1.36 $r(a) \in \sigma(\overline{a})^\wedge$. Therefore $r(a) \leq r(\overline{a})$. $\square$

Lemma 2.13 *Let I be a two-sided closed inessential ideal in the Banach algebra A. Then for every $a \in A$ the set $\sigma(a, A) \setminus \sigma(\overline{a}, A/I_A)$ is the union of the Riesz points of $\sigma(a)$ relative to I and some of the holes of $\sigma(\overline{a}, A/I_A)$.*

Proof: See Theorem 6.1 in [11].

$\square$

Theorem 2.14 *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. If $a \in C$ is such that $r(a)$ is a Riesz point of $\sigma(a)$, then psp(a) consists of Riesz points of $\sigma(a)$.*

Proof: Let $\lambda \in \mathrm{psp}(a)$. If $\lambda \in \sigma(\overline{a})$, then $r(a) = |\lambda| \leq r(\overline{a})$, so that $r(a) = r(\overline{a})$. But by Lemma 2.12 this is a contradiction with the fact that $r(a)$ is a Riesz point of $\sigma(a)$. Therefore $\mathrm{psp}(a) \subset \sigma(a) \setminus \sigma(\overline{a})$ and Lemma 2.13 now tells us that $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$. $\square$

Chapter 3

Poles of the resolvent in OBA's

In this chapter we investigate the role of poles of the resolvent in spectral theory. First we state versions of the Krein-Rutman Theorem in an OBA context following [18], then we take a closer look at the structure of the spectrum following [20].

3.1 Preliminaries

Lemma 3.1 *Let A be a Banach algebra and $a \in A$. If λ_0 is an isolated point of $\sigma(a)$ then*

$$(z-a)^{-1} = \sum_{n=-\infty}^{\infty} (z-\lambda_0)^n a_n$$

for $0 < |z-\lambda_0| < r_0 = d(\lambda_0, \sigma(a) \setminus \{\lambda_0\})$, where

$$a_n = \frac{1}{2\pi i} \int_\gamma (z-\lambda_0)^{-n-1}(z-a)^{-1} dz$$

for γ any positively oriented circle centered at λ_0 with radius $< r_0$.

The isolated point λ_0 is a pole of order $k \geq 1$ if and only if $a_{-k} \neq 0$ and $a_{-n} = 0$ for all $n > k$.

Proof: The first part is Lemma 6.11 in [10] and the series follows from the usual Laurent series development that can be found in Theorem 1.11 in [9]. The statement for the pole of order k follows from Corollary 1.18 in [9]. $\square$

Proposition 3.2 *Let A be a Banach algebra and $a \in A$. If λ_0 is an isolated point of $\sigma(a)$ and $n \geq 1$, then λ_0 is a pole of the resolvent function $R(z,a) = (z-a)^{-1}$ of order n if and only if $(\lambda_0 - a)^n p(a, \lambda_0) = 0$ and $(\lambda_0 - a)^{n-1} p(a, \lambda_0) \neq 0$.*

Proof: Let $(z-a)^{-1} = \sum_{n=-\infty}^{\infty} (z-\lambda_0)^n a_n$ as in Lemma 3.1. Now λ_0 is a pole of order n if and only if $a_{-n} \neq 0$ and $a_{-k} = 0$ for $k > n$. Let Γ be a positively oriented system of curves such that $\sigma(a) \setminus \{\lambda_0\} \subseteq \operatorname{ins} \Gamma$ and $\lambda_0 \in \operatorname{out} \Gamma$. Let γ be a circle centered at λ_0 and contained in out Γ. Let $e(z) \equiv 1$ in a neighborhood of $\gamma \cup \text{ins } \gamma$ and $e(z) \equiv 0$ in a neighborhood of $\Gamma \cup \text{ins } \Gamma$. So $e \in \text{Hol }(a)$ and $e(a) = p(a, \lambda_0)$. If $k \geq 1$,

$$\begin{aligned} a_{-k} &= \frac{1}{2\pi i} \int_\gamma (z - \lambda_0)^{k-1}(z-a)^{-1} dz \\ &= \frac{1}{2\pi i} \int_{\gamma+\Gamma} e(z)(z - \lambda_0)^{k-1}(z-a)^{-1} dz \\ &= p(a, \lambda_0)(a - \lambda_0)^{k-1}. \end{aligned}$$

The last stap follows from the functional calculus, since $\sigma(a) \subseteq \text{ins }(\gamma + \Gamma)$. The proposition follows.

□

Since $p(a, \lambda_0)$ is an idempotent it directly follows that:

Corollary 3.3 *Let A be a Banach algebra and $a \in A$. If λ_0 is an isolated point of $\sigma(a)$ and $n \geq 1$, then λ_0 is a pole of order n of the resolvent if and only if $(\lambda_0 - a)p(a, \lambda_0)$ is a nilpotent element of A of order n.*

3.2 Krein-Rutman Theorems

We will now state OBA versions of the Krein-Rutman Theorem, which is concerned with operators, following [18]. The Krein-Rutman Theorem describes conditions under which the spectral radius of a positive operator is an eigenvalue of that operator, with a positive eigenvector. For more information on this theorem we refer to [15].

First we state a version in which the condition that ensures that if a is positive, $r(a)$ is an eigenvalue of a with positive eigenvector, is in terms of $r(a)$.

Theorem 3.4 *Let A be an OBA with a closed algebra cone C and let $0 \neq a \in C$ be such that $r(a) > 0$. If $r(a)$ is a pole of the resolvent of a, then there exists $0 \neq u \in C$ such that $ua = au = r(a)u$ and $aua = r(a)^2 u$.*

Proof: Suppose that $r(a)$ is a pole of order k of the resolvent of a. Then we have according to Lemma 3.1 the following Laurent series development of the resolvent:

$$R(z, a) = \sum_{n=-k}^{\infty} (z - r(a))^n a_n, \quad 0 < |z - r(a)| < \text{dist}(r(a), \sigma(a) \setminus \{r(a)\}).$$

From the Laurent expression it follows that $a_{-k} = \lim_{z \downarrow r(a)} (z - r(a))^k R(z, a)$. We show that a_{-k} is a possible choice for u. It is clear that a commutes with a_{-k}. From the Neumann series $R(z, a) = \sum_{j=0}^{\infty} \frac{a_j}{z^{j+1}} \quad (z > r(a))$ for $R(z, a)$ and the fact that C is a closed algebra cone it follows that $R(z, a)$, and hence a_{-k}, is an element of C. From the proof of Proposition 3.2 it follows that $0 = a_{-(k+1)} = (r(a) - a)a_{-k} = a_{-k}(r(a) - a)$, which yields the first part of the theorem, with $u := a_{-k}$. Since $au = ua = r(a)u$, it follows that $aua = r(a)^2 u$. □

From the proof we see that if the pole $r(a)$ of the resolvent function is of order k, a possible choice for u is the coefficient a_{-k} from the Laurent series expression of the resolvent. To distinguish a_{-k} from possible other eigenvectors, we will call a_{-k} the (positive) Laurent eigenvector of the eigenvalue $r(a)$ of a.

From the proof we see that we have more generally:

Theorem 3.5 *Let A be a Banach algebra and $a \in A$. If α is a pole of the resolvent of a of order k, so that*

$$(z-a)^{-1} = \sum_{n=-k}^{\infty} (z-\alpha)^n a_n$$

and $0 \neq u := a_{-k}$, then $au = ua = \alpha u$.

Now we state another version of the Krein-Rutman theory in an OBA context.

Theorem 3.6 *Let A be a semisimple OBA with a closed normal algebra cone C and let $a \in C$ be such that $r(a) > 0$. If there exists a closed inessential ideal I in A such that a is Riesz w.r.t I, then there exists $0 \neq u \in C$ such that $ua = au = r(a)u$ and $aua = r(a)^2 u$.*

Before we can give the proof we need a few other theorems and lemmas.

Theorem 3.7 *Let A be a semisimple Banach algebra and I an inessential ideal of A. Then $I \subset \mathrm{kh}(\mathrm{soc}(A))$.*

Proof: This is Theorem 1.4 in [16]. □

From this theorem we get the following corollary.

Corollary 3.8 *Let A be a semisimple Banach algebra, $a \in A$ and I a closed inessential ideal of A. If a is Riesz relative to I then a is Riesz relative to $\mathrm{soc}(A)$.*

Proof: Suppose a is Riesz relative to I. According to Theorem 1.40 $\sigma(a)$ is finite or a sequence converging to zero, and for every $0 \neq \alpha \in \sigma(a)$ the spectral projection $p(a, \alpha)$ lies in I. By Theorem 3.7 we have $I \subset \mathrm{kh}(\mathrm{soc}(A))$, so that all these spectral projections are in $\mathrm{kh}(\mathrm{soc}(A))$. Corollary 1.33 tells us that $\mathrm{soc}(A)$ and $\mathrm{kh}(\mathrm{soc}(A))$ have the same projections, so it follows that all these spectral projections are in $\mathrm{soc}(A)$. Thus Theorem 1.40 implies that a is Riesz relative to $\mathrm{soc}(A)$. □

Lemma 3.9 *Let A be a semisimple Banach algebra and $a \in A$. If a is in $\mathrm{soc}(A)$ and a is quasinilpotent, then a is nilpotent.*

Proof: This is Lemma 3.10 in [18]. □

Theorem 3.10 *Let A be a semisimple Banach algebra, $a \in A$ and I a closed inessential ideal of A such that a is Riesz relative to I. If $0 \neq \alpha \in \sigma(a)$ then α is a pole of the resolvent of a.*

Proof: If a is Riesz relative to I, then by Corollary 3.8 we have that a is Riesz relative to $\mathrm{soc}(A)$. If $0 \neq \alpha \in \sigma(a)$, then it follows Theorem 1.40 that α

is an isolated point of $\sigma(a)$ and $p(a,\alpha)$ is in $\mathrm{soc}(A)$. Since $\mathrm{soc}(A)$ is an ideal, we have $(a-\alpha)p(a,\alpha) \in \mathrm{soc}(A)$. We know from ([8], Proposition 9, p36) that $(a-\alpha)p(a,\alpha)$ is quasinilpotent, so from Lemma 3.9 we see that $(a-\alpha)p(a,\alpha)$ is nilpotent. It follows from Corollary 3.3 that α is a pole of the resolvent of a. $\square$

In a similar way as for the previous theorem, we can prove the following related theorem. We do not use it to prove Theorem 3.6, but we will use it later on.

Theorem 3.11 *Let A be a semisimple Banach algebra, I an inessential ideal of A, and $a \in A$. Then a point α in $\sigma(a)$ is a Riesz point of $\sigma(a)$ relative to I if and only if α is a pole of the resolvent of a and $p(a,\alpha) \in I$.*

Proof: One implication is trivial. For the other implication let α be a Riesz point of $\sigma(a)$ relative to I. Then by definition α is an isolated point of $\sigma(a)$ and $p(a,\alpha) \in I$. From Theorem 3.7 and the fact that $\mathrm{kh}(\mathrm{soc}(A))$ and $\mathrm{soc}(A)$ have the same spectral projections (see Corollary 1.33) we see that $p(a,\alpha) \in \mathrm{soc}(A)$. Since $\mathrm{soc}(A)$ is an ideal, we have $(a-\alpha)p(a,\alpha) \in \mathrm{soc}(A)$. We know from ([8], Proposition 9, p36) that $(a-\alpha)p(a,\alpha)$ is quasinilpotent, so from Lemma 3.9 we see that $(a-\alpha)p(a,\alpha)$ is nilpotent. It follows from Corollary 3.3 that α is a pole of the resolvent of a. $\square$

Now we can give the proof of Theorem 3.6.

Proof: By Theorem 2.2 and Theorem 2.7, $r(a) \in \sigma(a)$. By assumption $r(a) \neq 0$, so by Theorem 3.10 $r(a)$ is a pole of the resolvent of a. The theorem now follows from Theorem 3.4. $\square$

3.3 More spectral theory

In this section we are going to investigate the influence that the structure of the spectrum $\sigma(a)$ has on some properties of a. We will follow [20]. First we discuss the case in which the spectrum consists of one element. Then we also consider spectra consisting of multiple elements.

The property of a we focus on is whether positivity of a implies that $a-1$ is positive, i.e. $a \geq 1$. Later on we discuss the more general case, if $f \in \mathrm{Hol}(a)$ and $f(a)$ defined by the functional calculus, whether $a \in C$ implies $f(a) \in C$.

Theorem 3.12 *Let (A,C) be an OBA with C closed and let $a \in C$. If $\lambda > r(a)$, then $(\lambda - a)^{-1} \geq 0$.*

Proof: For $|\lambda| > r(a)$, the resolvent of a has a Neumann series representation $(\lambda-a)^{-1} = \sum_{n=0}^{\infty}(a^n/\lambda^{n+1})$. Since $\lambda > 0$, all the terms in the series are positive, so because C is closed, we have $(\lambda-a)^{-1} \geq 0$. $\square$

Theorem 3.13 *Let A be a Banach algebra and $a \in A$ such that $\sigma(a) = \{\lambda_0\}$. If $\lambda \neq \lambda_0$, then*

$$(\lambda - a)^{-1} = \sum_{n=1}^{\infty} b_{-n}(\lambda - \lambda_0)^{-n}$$

where $b_{-n} = (a - \lambda_0)^{n-1}$

Proof: If $\lambda \neq \lambda_0$, then $|\lambda - \lambda_0| > 0 = r(a - \lambda_0)$, so that

$$(\lambda - a)^{-1} = ((\lambda - \lambda_0) - (a - \lambda_0))^{-1} = \sum_{n=0}^{\infty} \frac{(a - \lambda_0)^n}{(\lambda - \lambda_0)^{n+1}} = \sum_{n=1}^{\infty} \frac{(a - \lambda_0)^{n-1}}{(\lambda - \lambda_0)^n}.$$

Hence the result follows. □

The series above clearly is the Laurent series of the resolvent of a around λ_0, so we have

Theorem 3.14 *Let A be a Banach algebra and $a \in A$ such that $\sigma(a) = \{\lambda_0\}$. If λ_0 is a pole of order k of the resolvent of a, then $(a - \lambda_0)^k = 0$ and $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^k (\lambda - a)^{-1} = (a - \lambda_0)^{k-1}$.*

Now we can state some conditions which imply that if $a \in C$ and $\sigma(a) = \{r(a)\}$ with $r(a) \geq 1$, then $a - 1 \in C$.

Theorem 3.15 *Let A be a Banach algebra and $a \in A$ such that $\sigma(a) = \{r(a)\}$.*

1. *If $r(a)$ is a pole of order k of the resolvent of a, then $(a - r(a))^k = 0$.*
2. *If $r(a)$ is a simple pole of the resolvent of a, then $a = r(a)$. It follows that, if C is an algebra cone of A, then*

$$r(a) \geq 1 \Rightarrow a - 1 \in C$$

Suppose that C is a closed algebra cone of A, and $a \in C$.

3. *If $r(a)$ is a pole of order k of the resolvent of a, then $(a - r(a))^{k-1} \in C$.*
4. *If $r(a)$ is a pole of order 2 of the resolvent of a, then $a \geq r(a)$.*

Proof:

1. Follows directly from Theorem 3.14.
2. Follows from 1.
3. From Theorem 3.14 we have $(a - r(a))^{k-1} = \lim_{\lambda \to r(a)} (\lambda - r(a))^k (\lambda - a)^{-1}$, so we certainly have $(a - r(a))^{k-1} = \lim_{\lambda \to r(a)^+} (\lambda - r(a))^k (\lambda - a)^{-1}$. Since C is closed, it follows from Theorem 3.12 that $(a - r(a))^{k-1} \in C$.
4. Follows from 3.

□

Now we state some results about the following question: if $a \in C$, for which functions $f \in \mathrm{Hol}(a)$ does it follow that $f(a) \in C$?

Theorem 3.16 *Let (A, C) be an OBA and $a \in C$.*

1. *If $p(\lambda) = \alpha_n \lambda^n + \cdots + \alpha_1 \lambda_1 + \alpha_0$ with $\alpha_n, \cdots, \alpha_0$ real and positive, then $p(a) \in C$.*
2. *Suppose, in addition, that C is closed. If $f(\lambda) = e^{\lambda}$, then $f(a) \in C$.*

Proof: Follows from the functional calculus. $\square$

Theorem 3.17 *Let A be a Banach algebra and $a \in A$ such that $r(a)$ is a pole of order k of the resolvent of a. Let f be a complex valued function that is analytic in the open disk $D(r(a), R)$ for some $R > 0$. Let $g(\lambda) = f(\lambda)(\lambda - a)^{-1}$ and let $\sum_{n=-\infty}^{\infty} (\lambda - r(a))^n a_n$ be the Laurent series of g around $r(a)$.*

1. *If $f(r(a)) = 0$ and the order of f at $r(a)$ is k, then $a_{-1} = 0$.*

Suppose, in addition, that (A, C) is an OBA with C closed, $a \in C$ and $f(\lambda) > 0$ for all λ in the open real interval $(r(a), r(a) + R)$.

2. *If the order of f in $r(a)$ is equal to $j \geq 0$, then $a_{-k+j} \in C$ and $a_l = 0$ for $l \leq -k + j$.*

Proof:

1. If $f(r(a)) = 0$ and the order of f at $r(a)$ is k, then the order of g at $r(a)$ is zero, so its residue is zero. Hence $a_{-1} = 0$.
2. If the order of f in $r(a)$ is equal to $j \geq 0$, then the order of g at $r(a)$ is $k - j$, so $a_{-k+j} = \lim_{\lambda \to r(a)} (\lambda - r(a))^{k-j} g(\lambda)$. Restricting λ to the interval $(r(a), r(a)+R)$, we get $a_{-k+j} = \lim_{\lambda \to r(a)^+} (\lambda - r(a))^{k-j} f(\lambda)(\lambda - a)^{-1}$. For λ in $(r(a), r(a)+R)$ we have that $f(\lambda) > 0$ by assumption and $(\lambda - a)^{-1} \in C$ by Theorem 3.12, so $(\lambda - r(a))^{k-j} f(\lambda)(\lambda - a)^{-1} \in C$. Since C is closed, $a_{-k+j} \in C$. It is clear that $a_l = 0$ for $l < -k + j$.

$\square$

If we take $f = 1$ we know that a_{-1} is equal to the spectral projection $p(a, r(a))$, so that the above Theorem gives us.

Corollary 3.18 *Let (A, C) be an OBA with C closed, and $a \in C$ such that $r(a)$ is a simple pole of the resolvent of a, then $p(a, r(a)) \in C$.*

Theorem 3.19 *Let A be a Banach algebra and $a \in A$ such that $\sigma(a) = \{\lambda_1, \ldots, \lambda_m\}$ $(m \geq 1)$ where $\lambda_1 = r(a)$ and λ_j is a pole of order k_j of the resolvent of a $(j = 1, \ldots, m)$. Let $f \in \mathrm{Hol}(a)$, such f has a zero of order k_j at λ_j for $j = 2, \ldots, m$.*

1. *If $f(r(a)) = 0$ and the order of f at $r(a)$ is k_1, then $f(a) = 0$.*

Suppose, in addition, that (A, C) is an OBA with C closed, $a \in C$ and $f(\lambda) > 0$ in the real interval $(r(a), r(a) + b)$, for some $b > 0$.

3. *If order of f at $r(a)$ is $k_1 - 1$, then $f(a) \in C$.*

Proof: Let Γ be the union of circles with centers $\lambda_1, \ldots, \lambda_m$ and resp. radii $r_1, \ldots, r_m$ such that they are disjoint. Then the functional calculus gives us $f(a) = \frac{1}{2\pi i}\int_\Gamma g(\lambda)d\lambda = \sum_{j=1}^m \frac{1}{2\pi i}\int_{C(\lambda_j,r_j)} g(\lambda)d\lambda$, with $g(\lambda) = f(\lambda)(\lambda - a)^{-1}$. Since the order of g at λ_j is zero, it follows that $\int_{C(\lambda_j,r_j)} g(\lambda)d\lambda = 0$ for $j = 2, \ldots, m$, so that $f(a) = \frac{1}{2\pi i}\int_{C(\lambda_1,r_1)} g(\lambda)d\lambda$. We can choose the radius r_1 such that g is analytic in a deleted neighbourhood of $r(a)$ containing $C(r(a), r_1)$. Therefore $\frac{1}{2\pi i}\int_{C(\lambda_1,r_1)} g(\lambda)d\lambda$ is the residue of g at $r(a)$. So $f(a) = a_{-1}$, with a_{-1} the coefficient of $(\lambda - r(a))^{-1}$ in the Laurent series of g around $r(a)$. The results now follow from Theorem 3.17. □

We now give some corollaries of Theorem 3.19

Corollary 3.20 *Let A be a Banach algebra and $a \in A$ such that $r(a) = k\pi \in \sigma(a)$ with $k \in \mathbb{N}$ an even number, and*

$$\sigma(a) \setminus r(a) \subset \{n\pi : n \in \{0, \pm 1, \ldots, \pm k\}\}.$$

1. *If each value in $\sigma(a)$ is a simple pole of the resolvent of a, then $\sin a = 0$.*

Suppose, in addition, that (A, C) is an OBA with C closed, and $a \in C$.

2. *If each element of $\sigma(a) \setminus r(a)$ is a simple pole and $r(a)$ is a pole of order 2 of the resolvent of a, then $\sin a \in C$*

Proof: Let $f(\lambda) = \sin\lambda$. Then f has simple zeros at all the values of the spectrum of a and $f(\lambda) > 0$ for all λ in the real interval $(r(a), r(a) + \pi)$. Since $f(a) = \sin a$,

1. Follows from Theorem 3.19.1.

2. Follows from Theorem 3.19.3.

□

Corollary 3.21 *Let (A, C) be an OBA with C closed, and $a \in C$ such that $r(a) = (k + \frac{1}{2})\pi \in \sigma(a)$ with $k \in \mathbb{N}$ an even number, and*

$$\sigma(a) \setminus r(a) \subset \{n\pi : n \in \{0, \pm 1, \ldots, \pm k\}\}.$$

If each value in $\sigma(a)$ is a simple pole of the resolvent of a, then $\sin a \in C$.

Proof: Let $f(\lambda) = \sin\lambda$. Then f has simple zeros at all the values of $\sigma(a)\setminus r(a)$. Furthermore, $f(r(a)) = 1 > 0$ and $f(\lambda) > 0$ for all λ in the real interval $(r(a), r(a) + \frac{\pi}{2})$. Since $f(a) = \sin a$, the result follows from Theorem 3.19.2. □

Corollary 3.22 *Let A be a Banach algebra and $a \in A$ such that $\sigma(a) = \{r(a)\}$ with $r(a) > 0$.*

1. *If $r(a) = 1$ is a simple pole of the resolvent of a, then $\log a = 0$.*

Suppose, in addition, that (A, C) is an OBA with C closed, and $a \in C$

2. *If $r(a)$ is a simple pole of the resolvent of a and $r(a) > 1$, then $\log a \in C$.*

3. *If $r(a) = 1$ is a pole of order 2 of the resolvent of a, then $\log a \in C$.*

Proof: Let $f(\lambda) = \log \lambda$ ($\log \lambda$ is the principal branch of the complex logarithm), then f is analytic on the right half plane, so because $r(a) > 0$, $f \in \mathrm{Hol}(a)$. Also, f has a simple zero at 1, and $f(\lambda) > 0$ for all real $\lambda > 1$. Hence the results follow from Theorem 3.19. □

Corollary 3.23 *Let (A, C) be an OBA with C closed and $a \in C$ such that $\sigma(a) = \{1, r(a)\}$, with $r(a) > 1$. If both 1 and $r(a)$ are simple poles of the resolvent of a, then $\log a \in C$.*

Proof: Let $f = \log \lambda$, then as in the proof of the previous corollary we have $f \in \mathrm{Hol}(a)$ and $f(\lambda) > 0$ for all real $\lambda > 1$. Furthermore, 1 and $r(a)$ are both simple poles, hence the result follows from Theorem 3.19.2. □

Now we discuss the case of C being inverse-closed. First a theorem that complements Theorem 3.16 and 3.19.

Theorem 3.24 *Let (A, C) be an OBA with C inverse-closed, and $a \in C$. Let $p(\lambda) = \alpha_n\lambda^n + \cdots + \alpha_1\lambda + \alpha_0$ and $q(\lambda) = \beta_m\lambda^m + \cdots + \beta_1\lambda + \beta_0$ with $\alpha_n, \ldots, \alpha_0, \beta_m, \ldots, \beta_0$ real a positive. Suppose that $q(\lambda)$ has no zeroes in $\sigma(a)$ and let $t(\lambda) = p(\lambda)/q(\lambda)$. Then $t(a) \in C$.*

Proof: From Theorem 3.16.1 it follows that $p(a) \in C$ and $q(a) \in C$. According to the Spectral Mapping Theorem $\sigma(q(a)) = q(\sigma(a))$, and $q(\lambda)$ has no zeroes in $\sigma(a)$, so $q(a)$ is invertible and $q^{-1} \in \mathrm{Hol}(a)$. Since C is inverse-closed, $(q(a))^{-1} \in C$. From the functional calculus we have $t(a) = p(a)(q(a))^{-1}$, so it follows that $t(a) \in C$. □

Now we give some conditions under which it is true that $a \in C$ and $\sigma(a) = \{1\}$ imply that $a - 1 \in C$, under the assumption that C is inverse-closed.

We begin with an obvious lemma

Lemma 3.25 *Let (A, C) be an OBA with a and b invertible elements of A such that $a \le b$ and $a^{-1}, b^{-1} \ge 0$. Then $b^{-1} \le a^{-1}$.*

Theorem 3.26 *Let (A, C) be an OBA with C closed and inverse-closed. If $a \in C$ and a is invertible, then*

1. *$a \ge \alpha$ for all $\alpha \ge 0$ with $\alpha < \delta(a)$.*

2. *$a \le \beta$ for all $\beta > r(a)$.*

Proof:

1. For $\alpha = 0$ it is obviously true. Let $0 < \alpha < \delta(a)$, then $(1/\delta(a)) < (1/\alpha)$, so that $(1/\alpha) > r(a^{-1})$. It follows from Theorem 3.12 that $((1/\alpha) - a^{-1})^{-1} \ge 0$. Because C is inverse-closed $(1/\alpha) - a^{-1} \ge 0$, so we have $a^{-1} \le (1/\alpha)$. The result now follows from Lemma 3.25.

2. If $\beta > r(a)$, then according to Theorem 3.12, $(\beta - a)^{-1} \ge 0$. Since C is inverse-closed, it follows that $\beta - a \ge 0$, and hence $a \le \beta$.

$\square$

Theorem 3.27 *Let (A, C) be an OBA with C closed and inverse-closed, and let $a \in C$. Then we have*

1. $\delta(a) \leq a \leq r(a)$.

Suppose, in addition, C is proper. Then,

2. $\sigma(a) \subset \{z \in \mathbb{C} : |z| = 1\} \Rightarrow a = 1$.
3. $\sigma(a) = \{1\} \Rightarrow a = 1$.

Proof:

1. Let (α_n) be a sequence of real numbers such that $0 \leq \alpha_n < \delta(a)$ and $\alpha_n \to \delta(a)$ as $n \to \infty$. By Theorem 3.26.1, $a \geq \alpha_n$, i.e. $(a - \alpha_n) \in C$ for all n. Therefore $\lim_{n\to\infty}(a - \alpha_n) = a - \delta(a) \in C$, because C is closed. Let (β_n) be a sequence of real numbers such that $r(a) < \beta_n$ and $\beta_n \to r(a)$ as $n \to \infty$. Then $a \leq \beta_n$, by Theorem 3.26.2, so as before we have that $a \leq r(a)$.
2. If $\sigma(a) \subset \{z \in \mathbb{C} : |z| = 1\}$, then $\delta(a) = 1 = r(a)$, so by 1. we have that $1 \leq a \leq 1$. Therefore, because C is proper, it follows that $a = 1$.
3. Follows from 2.

$\square$

Lemma 3.28 *Let A be a Banach algebra and $a \in A$. If there exist $k \in \mathbb{N}$ and $\lambda_0 \in \mathbb{C}$ such that psp$(a^k) = \{\lambda_0\}$, then #psp$(a) \leq k$.*

Proof: If $\lambda \in \text{psp}(a)$, then (by the Spectral Mapping Theorem) $\lambda^k \in \text{psp}(a^k)$, so $\lambda^k = \lambda_0$. Hence every $\lambda \in \text{psp}(a)$ is a k-th root of λ_0 and thus $\#\text{psp}(a) \leq k$. $\square$

Theorem 3.29 *Let (A, C) be an OBA with C closed and the spectral radius function monotone. If $a \in A$ and there exist $k \in \mathbb{N}$ and $\alpha > 0$ such that $a^k \geq \alpha$, then*

1. psp$(a^k) = \{r(a)^k\}$.
2. #psp$(a) \leq k$.

Proof:

1. Since $\text{psp}(\beta a) = \beta \text{psp}(a)$ for all $\beta \geq 0$, we may assume without loss of generality that $r(a) = 1$. Let $b = a^k - \alpha$. Then $b \geq 0$. Since $a^k = b + \alpha$, it follows that $1 = r(a^k) = r(b + \alpha)$, so that $1 = \sup\{|\lambda + \alpha| : \lambda \in \sigma(b)\}$. Since $r(b) \in \sigma(b)$, by Theorem 2.7, this supremum is exactly $r(b) + \alpha$. Hence $r(b) = 1 - \alpha$, so that $\sigma(a^k) \subset \{\lambda + \alpha : |\lambda| \leq 1 - \alpha\}$.

   Now suppose $z \in \text{psp}(a^k)$. Then $z = \lambda + \alpha$ with $|\lambda| \leq 1 - \alpha$, so that $|z - \alpha| \leq 1 - \alpha$, and $|z| = 1$. Consequently $z \in \overline{D}(\alpha, 1 - \alpha) \cap \{z \in \mathbb{C} : |z| = 1\}$. Let $z = c + di$. Then $(c - \alpha)^2 + d^2 \leq (1 - \alpha)^2$ and $c^2 + d^2 = 1$, so that $2\alpha c \geq 2\alpha$, and hence $c \geq 1$, since $\alpha > 0$. Since $c^2 + d^2 = 1$, it follows that $c = 1$ and $d = 0$, so that $z = 1$. Hence the result follows.

2. Follows from 1. and Lemma 3.28.

□

Now with Theorem 3.26.1 and 3.29.1 we come to

Theorem 3.30 *Let (A, C) be an OBA with C closed, inverse-closed and the spectral radius function monotone. If $a \in C$ is an invertible element, then $psp(a) = \{r(a)\}$.*

Chapter 4

Representation theorems for OBA's

In this chapter we show that an OBA that satisfies certain conditions is isomorphic to the space of real-valued continuous functions $C_0(X)$ for a suitable compact Hausdorff space X. We make clear which space the space X is and state the results in a few representation theorems. We will follow [24] and [27].

4.1 Preliminaries

Let A be a Banach space. With A^* we denote the dual space of A and with wk* the weak-star topology of this space.

We state a corollary of the Hahn-Banach Theorem (see theorem 1.15).

Corollary 4.1 *If A is a normed space and $x \in A$, then*

$$\|x\| = \sup\{|f(x)| : f \in A^* \quad \textit{and } \|f\| \leq 1\}$$

Moreover, this supremum is attained.

Proof: This is Corollary 3.6.7 in [10]. □

If X is a normed space, denote by $\mathrm{ball}(X)$ the closed unit ball in X. So $\mathrm{ball}(X) := \{x \in X : \|x\| \leq 1\}$.

Theorem 4.2 (Alaoglu's Theorem) *If X is a normed space, then* $\mathrm{ball}(X^*)$ *is wk* compact.*

Proof: This is Theorem 5.3.1 in [10]. □

Now we define extreme points and state the Krein-Milman Theorem.

Definition 4.3 If K is a convex subset of a vector space X, then a point a in K is an extreme point of K if there is no proper open line segment that contains a and lies entirely in K. Let $\mathrm{ext}(K)$ be the set of extreme points of K.

An open line segment is a set of the form $(x_1, x_2) := \{tx_2 + (1-t)x_1\} : 0 < t < 1\}$, and this line segment is proper if $x_1 \neq x_2$.

Theorem 4.4 (The Krein-Milman Theorem) *If K is a nonempty compact convex subset of a locally convex space X, then ext(K) $\neq \emptyset$ and K is equal to the closed convex hull of ext(K).*

Proof: This is Theorem 5.7.4. in [10]. $\square$

4.2 Representation theorems for OBA's

Definition 4.5 Let A be a real algebra. We call a functional $f : A \to \mathbb{R}$ a *Schwarz map* if it satisfies the Schwarz inequality $f(a)^2 \leq f(a^2)$ for all $a \in A$. The set of all Schwarz maps is denoted by S_A. If A is ordered by an algebra cone C, we define S_A^+ to be the subset of all functionals with $f(C) \subset [0, \infty)$.

Lemma 4.6 *S_A and S_A^+ are convex.*

Proof: Let $f_1, f_2 \in S_A$, then

$$\begin{aligned}
(\frac{1}{2}f_1(a) + \frac{1}{2}f_2(a))^2 &= \frac{1}{4}f_1(a)^2 + \frac{1}{2}f_1(a)f_2(a) + \frac{1}{4}f_2(a)^2 \\
&\leq \frac{1}{4}f_1(a^2) + (\frac{1}{2}f_1(a) - \frac{1}{2}f_2(a))^2 + \frac{1}{2}f_1(a)f_2(a) + \frac{1}{4}f_2(a^2) \\
&\leq \frac{1}{2}f_1(a^2) + \frac{1}{2}f_2(a^2)
\end{aligned}$$

So for all $f_1, f_2 \in S_A$ we have $\frac{1}{2}f_1 + \frac{1}{2}f_2 \in S_A$. Therefore, given $t \in [0, 1]$, $\epsilon > 0$ and $f_1, f_2 \in S_A$, we can find a number s equal to $\frac{m}{2^n}$ fore some $m, n \in \mathbb{N}$, such that $|t - s| < \epsilon$ and $sf_1 + (1 - s)f_2 \in S_A$. So we can find a sequence of numbers s_n such that $s_n \to t$ as $n \to \infty$ and $(s_n f_1(a) + (1 - s_n)f_2(a))^2 \leq s_n f_1(a^2) + (1 - s_n)f_2(a^2)$ for all $a \in A$. Thus $(tf_1(a) + (1 - t)f_2(a))^2 \leq tf_1(a^2) + (1 - t)f_2(a^2)$ for all $a \in A$ and we have that $tf_1 + (1 - t)f_2 \in S_A$.

The convexity of S_A^+ follows from the convexity of S_A. $\square$

Let $f \in S_A$. If A is non-unital we define the map $\overline{f}$ on A_e as follows:

$$\overline{f}((a, \alpha)) := f(a) + \alpha \qquad (a, \alpha) \in A_e.$$

We have that

$$\begin{aligned}
\overline{f}((a, \alpha))^2 &= (f(a) + \alpha)^2 = f(a)^2 + 2\alpha f(a) + \alpha^2 \\
&\leq f(a^2) + 2\alpha f(a) + \alpha^2 = f(a^2 + 2\alpha a) + \alpha^2 \\
&= \overline{f}((a^2 + 2\alpha a, \alpha^2)) = \overline{f}((a, \alpha)^2),
\end{aligned}$$

hence $\overline{f}$ is a Schwarz map. If A is unital and we speak of $\overline{f}$, we just mean f. With the Schwarz inequality we see that a Schwarz map is non-negative on the squares. Conversely we have

Proposition 4.7 *Let f be a bounded linear functional on a real Banach algebra A, possessing a bounded approximate identity with norm bound L, which is non-negative on the squares. Then*

$$f(a)^2 \leq \|f\| L^2 f(a^2), \quad \forall a \in A.$$

Proof: Let $a, b \in A$, then for every $x \in \mathbb{R}$ we have

$$x^2 f(a^2) + x(f(ab) + f(ba)) + f(b^2) = f((xa + b)^2) \geq 0.$$

Using the formula for the discriminant for second degree equations we get

$$(f(ab) + f(ba))^2 \leq 4f(a^2)f(b^2). \tag{4.1}$$

Let $(e_\lambda)_{\lambda \in I}$ be a bounded approximated identity with norm bound L. Then,

$$f(e_\lambda^2) \leq \|f\|(\sup \|e_\lambda\|)^2 \leq \|f\|L^2.$$

Since f is continuous it follows from 4.1 that

$$(2f(a))^2 = \lim_{\lambda \in I}(f(ae_\lambda) + f(ae_\lambda))^2 \leq 4\|f\|L^2 f(a^2).$$

Thus,

$$f(a)^2 \leq \|f\|L^2 f(a^2).$$

□

Note that equation 4.1 is the weak Cauchy-Schwarz inequality.

Corollary 4.8 *Let f be a bounded (positive) linear functional on an unital Banach algebra A with $f(1) \leq 1$ which is non-negative on the squares. Then f is a (positive) Schwarz map on A.*

Proof: Putting $b = 1$ in equation (4.1) we get the result. □

Lemma 4.9 *Let A be a real Banach algebra, not necessarily unital. Then every Schwarz map is continuous and satisfies $|f(a)| \leq r(a)$ for all $a \in A$.*

Proof: Using the functional calculus we see that for all $a \in A_e$ with $r(a) < 1$ there exists $b \in A_e$ such that $(1 - a) = b^2$. If $A = A_e$, then $0 \leq f(b^2) = f(1) - f(a)$, so $f(a) \leq f(1) \leq 1$. If A is non-unital, then $0 \leq \overline{f}(b^2) = 1 - f(a)$, so again $f(a) \leq 1$. Now we replace a by $a^2/(r(a)^2 + \epsilon)$ with $\epsilon > 0$ and get $f(a)^2 \leq f(a^2) \leq r(a)^2 + \epsilon$. Hence $|f(a)| \leq r(a)$. Since $r(a) \leq \|a\|$, f is continuous. □

From now on we will denote ball(A^*) with Σ, so $\Sigma := \{f \in A^* : \|f\| \leq 1\}$.

Lemma 4.10 *S_A, S_A^+ and Σ^+ are wk*-compact.*

Proof: It follows from the preceding lemma that $\|f\| \leq 1$, hence $S_A \subset \Sigma$. The set Σ is wk*-compact by Alaoglu's Theorem. So if S_A is wk*-closed in Σ we are done. Let g_i be a net in S_A and $f \in \Sigma$ such that $g_i \overset{wk^*}{\to} f$. Then for all i we have $g_i(a)^2 - g_i(a^2) \leq 0$, and therefore $f(a)^2 - f(a^2) \leq 0$. Thus $f \in S_A$ and S_A is closed in Σ.

Now we show that S_A^+ is w*-closed in S_A. Let g_i be a net in S_A^+ and $f \in S_A$ such that $g_i \overset{wk^*}{\to} f$. Then for $a \in C$, for all i, we have $g_i(a) \geq 0$, so $f(a) \geq 0$. Thus $f \in S_A^+$ and S_A^+ is closed in S_A.

Similarly we can show that Σ^+ is compact. $\square$

Note that it follows from Proposition 4.7 that, if A possesses a bounded approximate identity with norm bound 1 and C contains all squares, then $\Sigma^+ \subset S_A^+$, thus $\Sigma^+ = S_A^+$.

Now we turn our attention to the positive multiplicative functionals.

Definition 4.11 Let M denote the set of positive multiplicative functionals:

$$M = \{f \in \Sigma^+ : f(ab) = f(a)f(b),\ \forall a, b \in A\}.$$

Lemma 4.12 *M is wk*-compact.*

Proof: Again we show that M is wk*-closed in the compact set Σ^+. Let g_i be a net in M and $f \in \Sigma^+$ such that $g_i \stackrel{wk^*}{\to} f$. Then for all i we have $g_i(a)g_i(b) - g_i(ab) = 0$, thus $f(a)f(b) - f(ab) = 0$. So $f \in M$ and M is closed in Σ^+.

$\square$

For our main theorem we need one more lemma.

Lemma 4.13 *A multiplicative Schwarz map is an extreme point of S_A*

Proof: Let f be a multiplicative Schwarz map. Suppose that $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, with $f_1, f_2 \in S_A$. Then we have

$$f_i(a)^2 \leq f_i(a^2), \quad (i = 1, 2).$$

Since f is multiplicative we also have $f(a^2) = f(a)^2$, which leads to,

$$\frac{1}{2}(f_1(a)^2 + f_2(a)^2) \leq \frac{1}{2}(f_1(a^2) + f_2(a^2)) = f(a^2) = f(a)^2 = \frac{1}{4}(f_1(a)^2 + 2f_1(a)f_2(a) + f_2(a)^2).$$

From this inequality it follows that $(f_1(a) - f_2(a))^2 \leq 0$, so $f_1(a) = f_2(a)$. The element a was arbitrary, so $f_1 = f_2$ and therefore f is an extreme point of S_A.
$\square$

Now we consider the following two conditions

(1) $\overline{f}(x^2a^2) \geq 0$ for every $f \in S_A^+$ and for all $x, a \in A_e$.

(2) $x^2a \in C$ for all $x \in A_e$ and all $a \in C$.

Definition 4.14 Let F denote the set of extreme points of S_A^+.

Theorem 4.15 *Let A be a real Banach algebra with a cone satisfying the conditions (1) and (2). Then the extreme points of S_A^+ are exactly the positive multiplicative functionals, i.e. $F = M$.*

Proof: Let $f \in S_A^+$ be extreme. If A possesses a unit element put $\overline{f} = f$, otherwise consider $\overline{f} : A_e \to \mathbb{R}$. For $x \in A_e$ with $r(x) < 1$ define $S_{x^2} : A_e \to \mathbb{R}$ by

$$S_{x^2}(a) := \overline{f}(x^2a) - \overline{f}(x^2)\overline{f}(a).$$

We show that $f_\pm := \overline{f} \pm S_{x^2}$ is non-negative on squares and positive. We have

$$f_+(a) = (1 - \overline{f}(x^2))\overline{f}(a) + \overline{f}(x^2a).$$

From Lemma 4.9 it follows that $\overline{f}(x^2) \leq r(x^2) = r(x)^2 < 1$. Therefore we have $1 - \overline{f}(x^2) > 0$. Thus f_+ is positive by condition (2) and non-negative on squares of A_e by condition (1). We know that $1 - x^2 = b^2$ for some $b \in A_e$, so

$$f_-(a) = \overline{f}(a - x^2a) + \overline{f}(x^2)\overline{f}(a) = \overline{f}(b^2a) + \overline{f}(x^2)\overline{f}(a).$$

is positive and non-negative on squares of A_e by conditions (1) and (2). Since $1 - x^2 = b^2$ and $\overline{f}$ is linear we have $f_\pm(1) = 1$. Therefore $f_\pm$ are positive Schwartz maps on A_e by Proposition 4.7 and the restrictions of $f_\pm$ to A are in S_A^+. We have $f(a) = \frac{1}{2}f_+(a) + \frac{1}{2}f_-(a)$ for all $a \in A$, so because f is extreme, $S_{x^2}(a) = 0$, for all $a \in A$. This implies that $f(x^2a) = \overline{f}(x^2)f(a)$ for all $a \in A$ and for all $x \in A_e$ with $r(x) < 1$, thus for all $x \in A_e$ by the linearity of f. Let $a, b \in A$. We may assume that $\|b\| < 1$ because of the linearity of f. Hence $1 - b = x^2$ for some $x \in A_e$ and we have

$$\begin{aligned} f(a) - f(ba) &= f(a - ba) = \overline{f}(a - ba) = \overline{f}((1-b)a) = \overline{f}(x^2a) = f(x^2a) = \overline{f}(x^2)f(a) \\ &= \overline{f}(1-b)f(a) = (1 - f(b))f(a) = f(a) - f(b)f(a), \end{aligned}$$

from which it follows that $f(ba) = f(b)f(a)$. From Lemma 4.13 it follows that a multiplicative positive Schwarz map is always an extreme point of S_A^+ and we are done. ☐

Lemma 4.16 *Let A be a Banach algebra possessing a bounded approximate identity with norm bound* 1. *Then a closed algebra cone C containing all squares satisfies conditions (1) and (2). For every bounded positive linear functional f there exists $\lambda > 0$ such that λf is a Schwarz map.*

Proof: Let $(e_\lambda)_{\lambda \in I}$ be a bounded approximate identity with norm bound 1 and let $x, a \in A_e$. Then we have $(xe_\lambda)^2 \in C$ and $(ae_\lambda)^2 \in C$. So $(xe_\lambda)^2(ae_\lambda)^2$ is positive, thus $f((xe_\lambda)^2(ae_\lambda)^2) \geq 0$. Let $x', a' \in A$ and $\beta, \gamma \in \mathbb{R}$ such that $x = (x', \beta)$ and $a = (a', \gamma)$. If we then work out the products $(xe_\lambda)^2(ae_\lambda)^2$ and x^2a^2 we see from the continuity and linearity of f that $f((xe_\lambda)^2(ae_\lambda)^2 - \beta^2\gamma^2e_\lambda^2)$ converges to $\overline{f}(x^2a^2 - \beta^2\gamma^2)$. Since $\|e_\lambda\| \leq 1$, we have $f(\beta^2\gamma^2e_\lambda^2) \leq \beta^2\gamma^2$. Therefore,

$$\begin{aligned} 0 &\leq f((xe_\lambda)^2(ae_\lambda)^2) \\ &= f((xe_\lambda)^2(ae_\lambda)^2 - \beta^2\gamma^2e_\lambda^2) + f(\beta^2\gamma^2e_\lambda^2) \\ &\leq f((xe_\lambda)^2(ae_\lambda)^2 - \beta^2\gamma^2e_\lambda^2) + \beta^2\gamma^2 \\ &\to \overline{f}(x^2a^2 - \beta^2\gamma^2) + \beta^2\gamma^2 = \overline{f}(x^2a^2). \end{aligned}$$

Hence $\overline{f}(x^2a^2) \geq 0$ and condition (1) is satisfied. Condition (2) follows directly from $x^2a = \lim_\lambda (xe_\lambda)^2a$.

Let $f : A \to \mathbb{R}$ be a continuous positive linear functional. Then

$$f(e_\lambda^2) \leq \|f\| \sup \|e_\lambda\|^2 \leq \|f\|.$$

We also have $2f(a) = \lim_{\lambda \in I} f(ae_\lambda + e_\lambda a)$ and from the weak Cauchy-Schwartz inequality (4.1) it follows that $4f(a)^2 \leq 4\|f\| f(a^2)$ for all $a \in A$. Hence λf is a Schwartz map for $\lambda = 4/\|f\|$.

$\square$

Let A^* be the Banach dual space of A, with dual cone C^* defined by

$$C^* = \{f \in A^* : f(a) \geq 0, \forall a \in C\}$$

We will need the following consequence of the Hahn-Banach Theorem.

Lemma 4.17 *If K is a closed convex set in a real Banach space A, and $x \notin K$, then there is an $f \in A^*$ with $f(x) < f(y)$ for all $y \in C$.*

Lemma 4.18 *If A is a real Banach space and C a closed cone in A and C^* the dual cone, then $x \in C$ if and only if $f(x) \geq 0$ for all $f \in C^*$. Also, $C \neq A$ implies that $C^* \neq \{0\}$.*

Proof: Suppose $x \notin C$. Since C is closed and convex it follows from Lemma 4.17 that there exists $f \in A^*$ with $f(x) < f(y)$ for all $y \in C$. So $f(x) < 0 = f(0)$. Suppose that there is a $y \in C$ such that $f(y) < 0$, then there is a $\lambda > 0$ such that $\lambda f(y) < f(x)$. But since f is linear and $\lambda y \in C$ this is a contradiction. Thus $f(y) \geq 0$ for all $y \in C$, hence $f \in C^*$. $\square$

With this lemma we see that $C = \{a \in A : f(a) \geq 0,\ \forall f \in C^*\}$ if C is a closed cone. We say that C and C^* are compatible.

Definition 4.19 The dual cone C^* is said to be *α-generated* if each $f \in A^*$ has a decomposition $f = f_1 - f_2$ with $f_1, f_2 \in C^*$ and

$$\alpha\|f\| \geq \|f_1\| + \|f_2\|.$$

If C^* is 1-generated then it follows from the triangle inequality that each $f \in A^*$ has a *Jordan decomposition*, that is, $f = f_1 - f_2$ with $f_1, f_2 \in C^*$ and

$$\|f\| = \|f_1\| + \|f_2\|.$$

The following proposition restates Grosberg and Krein's result on the equivalence of α-normality of C and α-generation of C^* stated in [12].

Proposition 4.20 *Let (A, C) be an OBA, then C is α-normal if and only if C^* is α-generated. In particular, C is 1-normal if and only if each $f \in A^*$ has a Jordan decomposition.*

Proof: This is Proposition 1.2 in [28] $\square$

Let X be a locally compact Hausdorff space. By $C_0(X)$ we denote the Banach algebra of all real-valued continuous functions on X vanishing at infinity. Let

$$C_0^+(X) = \{f \in C_0(X) : f(x) \geq 0, \forall x \in X\},$$

then $C_0^+(X)$ is a algebra cone and makes $C_0(X)$ an OBA.

Now we come to the following theorem.

Theorem 4.21 *Let (A, C) be a real OBA with C closed and 1-normal. Then A is isometric and algebraic-order isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, if and only if C contains all squares and A possesses a bounded approximated identity with norm bound 1.*

Proof: First we prove the necessity, From the definition of the cone $C_0^+(X)$ and the fact that $f(x)^2 \geq 0$ we see that this cone contains all squares. From Urysohn's Lemma it follows that $C_0(X)$ has bounded approximated identity with norm bound 1.

Now we prove the sufficiency. The set F from definition 4.14 is wk*-compact by Lemma 4.16, Theorem 4.15 and Lemma 4.12 . Let $X = F \setminus \{0\}$, with the wk*-topology. Suppose $C = A$, then $0 \leq a \leq b$ is valid for all $a, b \in A$. Since C is normal this cannot be the case, thus $C \neq A$, and $C^* \neq \{0\}$ by Lemma 4.18. It follows from Lemma 4.16 that $S_A^+ \neq \{0\}$, so $X \neq \emptyset$. Then X is a locally compact Hausdorff space. We define a map $\zeta : a \mapsto \widehat{a}$ from A into $C(X)$ by

$$\widehat{a}(f) = f(a), \quad \forall f \in X.$$

Since the maps in X are multiplicative linear functionals, ζ is an algebra homomorphism.

It is also an order-isomorphism. Let $a, b \in A$ with $a \leq b$. Then $\zeta(b-a)(f) = f(b-a) \geq 0$ for all $f \in X$ and we see that $\zeta(b) - \zeta(a) \in C^+(X, \mathbb{R})$, i.e. $\zeta(a) \leq \zeta(b)$. Conversely, let $a, b \in A$ with $\zeta(a) \leq \zeta(b)$. Then $f(b-a) \geq 0$ for all $f \in X$. Theorem 4.15 and the Krein-Milman Theorem show that S_A^+ is the wk*-closed convex hull of F and from Lemma 4.16 it also follows that S_A^+ generates the dual cone C^*. So $f(b-a) \geq 0$ for all $f \in C^*$ and we have that $b - a \in C$ by Lemma 4.18.

Now we show that ζ is an isometry. Since S_A^+ is the wk*-closed convex hull of F, we have

$$\|\widehat{a}\| = \sup\{|f(a)| : f \in X\} = \sup\{|f(a)| : f \in S_A^+\}.$$

The Hahn-Banach theorem tells us that

$$\|a\| = \sup\{|f(a)| : f \in \Sigma\}.$$

Recall from the remark following Lemma 4.10 that $\Sigma^+ = S_A^+$, because C contains all squares. We are now going to show that

$$\|a\| = \sup\{|f(a)| : f \in S_A^+\}.$$

Let $s = \sup\{|f(a)| : f \in S_A^+\}$. Since C is 1-normal, it follows from Propos- tion 4.20 that for $f \in \Sigma$ there exist $f_1, f_2 \geq 0$ with $\|f_1\| + \|f_2\| = \|f\|$ such that $f = f_1 - f_2$. Assuming $f_1, f_2 \neq 0$, we have that $\|f_1\|^{-1} f_1, \|f_2\|^{-1} f_2 \in S_A^+$. So

$$|f(a)| = |f_1(a) - f_2(a)| \leq \|f_1\| |\|f_1\|^{-1} f_1(a)|$$
$$+ \|f_2\| |\|f_2\|^{-1} f_2(a)| \leq \|f_1\| s + \|f_2\| s = \|f\| s \leq s.$$

If either $f_1 = 0$ or $f_2 = 0$ the resulting inequality is trivial. Thus we have

$$\|a\| = \sup\{|f(a)| : f \in \Sigma\} \leq s.$$

It is obvious that

$$\|a\| = \sup\{|f(a)| : f \in \Sigma\} \geq \sup\{|f(a)| : f \in S_A^+\} = s.$$

Therefore it follows that $\|a\| = \sup\{|f(a)| : f \in S_A^+\}$ and we have

$$\|\widehat{a}\| = \sup\{|\widehat{a}(f)| : f \in X\} = \sup\{|f(a)| : f \in S_A^+\} = \|a\|.$$

From this result we see that the map ζ is an isometry. Let $\widehat{A} = \{\widehat{a} : a \in A\}$. Let $\widehat{a} \in \widehat{A}$ and $\epsilon > 0$. Suppose $f \in X$ with $|\widehat{a}(f)| = |f(a)| \geq \epsilon$, then $f \neq 0$. So the set $\{f \in X : |\widehat{a}(f)| \geq \epsilon\}$ is equal to $\{f \in F : |\widehat{a}(f)| \geq \epsilon\}$ and therefore is compact, since it is a closed subset of the compact set F. Thus $\widehat{A}$ is a subalgebra of $C_0(X)$. Let $f, g \in X$ with $f \neq g$, then there exists an $a \in A$ such that $\widehat{a}(f) = f(a) \neq g(a) = \widehat{a}(g)$. So $\widehat{A}$ separates the points in X. Finally, for $f \in X$ there exists $a \in A$ such that $\widehat{a}(f) = f(a) \neq 0$, because $f \neq 0$. By Theorem 1.13, $\widehat{A}$ is dense in $C_0(X)$. Since A is complete and ζ is an isometry, it follows that $\widehat{A}$ is complete. Hence $\widehat{A} = C_0(X)$. □

From this theorem we directly have the following result.

Corollary 4.22 *Let (A, C) be a real OBA with C closed and 1-normal such that C contains all squares and A possesses a bounded approximated identity with norm bound 1. Then A is commutative.*

Now we prove a theorem stated by Kung-fu Ng in [24] using preceding lemma's.

Theorem 4.23 *Let (A, C) be a real OBA with C 1-normal and closed. Then A is isometrically and algebraically order isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, if and only if the following conditions are satisfied:*

(1) For $a, b \in A$ with $\|a\| = 1 = \|b\|$, there exists a $c \in A$ with $\|c\| = 1$ such that $0, a, b \leq c$.

(2) If $a, b \geq 0$ and $\|a\| = 1 = \|b\|$, then $ab \leq a, b$.

(3) For each $c \in C$ there exists two sequences (a_n) and (b_n) of positive elements with $\|a_n\| \leq 1$ and $\|b_n\| \leq 1$ such that $\lim_n ca_n = c = \lim_n b_n c$.

Proof: We have the following lemmas.

Lemma 4.24 *The ordered Banach algebra $C_0(X)$ satisfies conditions (1)-(3).*

Proof: Let $f_1, f_2 \in C_0(X)$ with $\|f_1\| = 1 = \|f_2\|$. Let $f_3 := \max(|f_1|, |f_2|)$ then $f_3 \in C_0(X)$, $\|f_3\| = 1$ and $0, f_1, f_2 \leq f_3$. So (1) is satisfied. If $f_1, f_2 \in C_0^+(X)$ and $\|f_1\| = \|f_2\| = 1$, then $f_1(1 - f_2)(x) \geq 0$ for all $x \in X$, so $f_1 f_2 \leq f_1$ and likewise $f_1 f_2 \leq f_2$. So (2) is satisfied. To show that (3) is also satisfied, let $f \in C_0^+(X)$. For $n \in \mathbb{N}$ we have

$$nf \cdot (1 + nf)^{-1} \in C_0^+(X) \text{ and } \|nf \cdot (1 + nf)^{-1}\| \leq 1.$$

Also, $0 \leq f(x) \cdot (1 + nf(x))^{-1} < 1/n$ for each $x \in X$ and $n \in N$. Therefore

$$\begin{aligned} \|f - f \cdot (nf \cdot (1 + nf)^{-1})\| &= \|f \cdot (1 + nf)^{-1}\| \\ &= \sup\{f(x)(1 + nf(x))^{-1} : x \in X\} \leq 1/n \end{aligned}$$

It follows that $\lim_n(f \cdot nf(1+nf)^{-1}) = f = \lim_n(nf(1+nf)^{-1} \cdot f)$. $\square$

This lemma proofs the necessity part of the theorem. Now we prove the sufficiency part. We first show that A has an approximate identity.

Lemma 4.25 *Let (A, C) be an OBA satisfying conditions (1)-(3) and C normal, then A has an bounded approximated identity of norm bound at most one.*

Proof: Suppose A satisfies (1)-(3). Let α be the normality constant. Let Λ be the set of all positive elements λ with $\|\lambda\| = 1$, and let $e_\lambda = \lambda$. From (1) we see that Λ is directed by $\leq$, so $\{e_\lambda, \lambda \in \Lambda, \leq\}$ is a net. We will show that

$$\lim ae_\lambda = a, \quad \forall a \in A. \tag{4.2}$$

First we show that

$$\lim ae_\lambda = a, \quad \forall a \in A \text{ with } a \geq 0 \tag{4.3}$$

Let $a \geq 0$. If $a = 0$ the equality is trivial. Suppose $a \neq 0$. We may assume that $\|a\| = 1$. Let $0 < \epsilon < 1$. By (3) there exists a $b \geq 0$ with $\|b\| \leq 1$ such that $\|a - ab\| < \epsilon$. Clearly $b \neq 0$. Let $\lambda_0 = \|b\|^{-1}b$ and suppose that $\lambda \in \Lambda$ with $\lambda \geq \lambda_0$. Then $\lambda \geq \|b\|^{-1}b \geq b$, so $a(\lambda - b) \in C$ and thus $a - ae_\lambda = a - a\lambda \leq a - ab$. By (2) we have that $ae_\lambda \leq a$, so $0 \leq a - ae_\lambda$ and therefore

$$\|a - ae_\lambda\| \leq \alpha\|a - ab\| < \alpha\epsilon, \quad \forall \lambda \in \Lambda, \quad \lambda \geq \lambda_0$$

Since ϵ was arbitrary, (4.3) is proved.

Now let $a \in A$. The case $a = 0$ is trivial. Suppose $a \neq 0$. By (1) there is a $b \geq 0$ such that $-\|a\|^{-1}a, \|a\|^{-1}a \leq b$. Write $a = a_1 - a_2$ with $a_1 = \frac{1}{2}(\|a\|b + a)$ and $a_2 = \frac{1}{2}(\|a\|b - a)$, then $a_1, a_2 \geq 0$. So it follows from (4.3) that

$$\lim a_i e_\lambda = a_i \quad (i = 1, 2).$$

Thus,

$$\lim ae_\lambda = \lim((a_1 - a_2)e_\lambda) = \lim a_1 e_\lambda - \lim a_2 e_\lambda = a_1 - a_2 = a.$$

and (4.2) is proved. In the same way we can show that

$$\lim e_\lambda a = a, \quad \forall a \in A. \tag{4.4}$$

$\square$

Now we will prove that C contains all the squares.

Let $\Sigma = \{f \in A^* : \|f\| \leq 1\}$ then Σ is wk*-compact by Alaoglu's Theorem. It is also true that C^* is wk*-closed. Let $f \in (C^*)^c$, then there exists $a \in C$ such that $f(a) = \delta < 0$. Let $g \in A^*$ such that $|f(a) - g(a)| < |\delta|$, then $g(a) < 0$ and thus $g \in (C^*)^c$. So $(C^*)^c$ is wk*-open and therefore C^* is wk*-closed. Let $\Sigma^+ = \Sigma \cap C^*$, then Σ^+ is wk*-compact and Σ^+ is convex. Let F' be the set of all extreme points of Σ^+. Then it follows from the Krein-Milman theorem that Σ^+ is the wk*-closed convex hull of F'.

We need the following lemmas:

Lemma 4.26 *If f is a positive bounded linear functional on A and $\{e_\lambda, \lambda \in \Lambda, \leq\}$ the net of all positive elements with norm 1 (as in the proof Lemma 4.25), then*

$$\|f\| = \lim f(e_\lambda). \tag{4.5}$$

Proof: By (1) Since $\|e_\lambda\| = 1$ for all $\lambda \in \Lambda$, we have $f(e_\lambda) \leq \|f\|$. Now let $\epsilon > 0$. Let $a \in A$ with $\|a\| = 1$ such that $\|f\| - \epsilon < f(a)$. By (1), there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq a$. Since f is positive, we have that

$$\|f\| - \epsilon < f(a) \leq f(e_\lambda), \quad \forall \lambda \in \Lambda,\ \lambda \geq \lambda_0.$$

Since ϵ was arbitrary, (4.5) is proved. □

Lemma 4.27 *Let $f \in F'$, then $f(ab) = f(a)f(b)$ for all $a, b \in A$.*

Proof: Let $f \in F'$. If $f = 0$, clearly f is multiplicative. Suppose $f \neq 0$. We have $0, \|f\|^{-1}f \in \Sigma^+$ and $f = \|f\|(\|f\|^{-1}f) + (1 - \|f\|)(0)$, so since f is an extreme point of Σ^+ it follows that $\|f\| = 1$. Next we show that for all $a \in C$,

$$f(ab) = f(a)f(b), \quad \forall b \in A. \tag{4.6}$$

For $a = 0$ the equation is trivial. Suppose $a \neq 0$. We may assume that $\|a\| = 1$. Let $f_1(b) = (f(b) + f(ab))(1 + f(a))^{-1}$. Then $f_1 \geq 0$. Also it follows from equations (4.5) and (4.2) and the fact that f is continuous that

$$\|f_1\| = \lim f_1(e_\lambda) = \lim(f(e_\lambda)+f(ae_\lambda))(1+f(a))^{-1} = (\|f\|+f(a))(1+f(a))^{-1} = 1.$$

Therefore $f_1 \in \Sigma^+$. Let $f_2 = 2f - f_1$, so

$$f_2(b) = (f(b) + 2f(a)f(b) - f(ab))(1 + f(a))^{-1}, \quad \forall b \in A. \tag{4.7}$$

If $b \geq 0$, it follows from (2) that $ab \leq b$ and because f is positive we have $f(ab) \leq f(b)$. So from (4.7) we see that $f_2(b) \geq 0$, i.e. f_2 is positive. Again, it follows from equations (4.5) and (4.2) and the fact that f is continuous that

$$\begin{aligned} \|f_2\| = \lim f_2(e_\lambda) &= \lim(f(e_\lambda) + 2f(a)f(e_\lambda) - f(ae_\lambda))(1 + f(a))^{-1} \\ &= (\|f\| + 2f(a)\|f\| - f(a))(1 + f(a))^{-1} = 1. \end{aligned}$$

Therefore $f_2 \in \Sigma^+$. Because $f = \frac{1}{2}(f_1 + f_2)$ and since f is an extreme point of S_A^+, we have that $f = f_1 = f_2$. So $f(b) = (f(b)+f(ab))(1+f(a))^{-1}$, from which 4.6 follows. Now let $a \in A$. By (1), we can write $a = a_1 - a_2$ with $a_1, a_2 \geq 0$ and it follows from 4.6 that

$$f(a_i b) = f(a_i)f(b) \quad (i = 1, 2).$$

Hence,

$$\begin{aligned} f(ab) &= f(a_1 b) - f(a_2 b) = f(a_1)f(b) - f(a_2)f(b) \\ &= (f(a_1) - f(a_2))f(b) = f(a)f(b), \quad \forall b \in A. \end{aligned}$$

□

Lemma 4.28 *Let (A, C) be an OBA satisfying conditions (1)-(3), then C contains all squares.*

Proof: By the preceding lemma, we have $f(a^2) = f(a)^2 \geq 0$ for all $f \in F'$. We already saw with the Krein-Milman Theorem, that Σ^+ is the wk*-closed convex hull of F'. Therefore $f(a^2) \geq 0$ for all f in Σ^+ and therefore for all $f \in C^*$. Since C and C^* are compatible, it follows that $a^2 \geq 0$. □

Now we can apply Theorem 4.21 in order to prove Theorem 4.23 and we see that A is isometrically and algebraically order-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

□

If the a real OBA has a unit element, we may assume with norm 1, we have the following corollary:

Corollary 4.29 *Let (A, C) be a real OBA with unit element of norm one and C 1-normal and closed. Then A is isometrically and algebraically order-isomorphic to $C(X)$ for some compact Hausdorff space X, if and only if it satisfies,*

(1) For $a, b \in A$ with $\|a\| = 1 = \|b\|$, there exists a $c \in A$ with $\|c\| = 1$ such that $0, a, b \leq c$.

(2) If $a, b \geq 0$ and $\|a\| = 1 = \|b\|$, then $ab \leq a, b$.

Proof: By definition e is positive, so A satisfies condition (3) from Theorem 4.23, since we can take $b_n = c_n = e$. Now we can apply Theorem 4.23 and it follows that A can be represented by $C_0(X)$ for some locally compact Hausdorff space X. Because A is algebraically isomorphic to $C_0(X)$ and A has a algebraic identity, $C_0(X)$ has a multiplicative identity, say $\mathbf{1}$. Urysohns Lemma tells us that for every $x_0 \in X$ there exists a $f \in C_0(X)$ with $f(x_0) \neq 0$. So because the multiplication in $C_0(X)$ is defined pointwise it follows that the multiplicative identity must be constant function 1. By the definition of $C_0(X)$ the set $\{x \in X : |\mathbf{1}(x)| > 1/2\}$ is compact. Hence X is compact and we are done. □

Now we have three examples to show that conditions $(1), (2), (3)$ are independent.

Example 4.30 *Let A be the set of all pairs (α, β) of real numbers with the operations defined coordinatewise, $\|(\alpha, \beta)\| = \max\{|\alpha|, |\beta|\}$ and $C = \{(\alpha, \beta) \in A : \alpha \geq 0, \beta = 0\}$. Then A is an OBA, with identity of norm 1 and C 1-normal and closed, that satisfies conditions (2) and (3), but not condition (1). Moreover, the identity map of A onto $\mathbb{R}^2$ is an isometric isomorphism, but not an order-isomorphism.*

Proof: Direct calculation shows that (A, C) is an OBA, the identity has norm 1 and C is 1-normal. It is not hard to verify that A satisfies (2), and A does not satisfy (1). Since $(1, 0)$ is an algebraic identity for positive elements, A satisfies (3). The last statement is easy to verify. □

Example 4.31 *Let A be the set of all pairs (α, β) of real numbers with the operations defined coordinatewise, $\|(\alpha, \beta)\| = \frac{1}{2}\max\{|\alpha|, |\beta|\}$ and $C = \{(\alpha, \beta) \in$*

$A : \alpha \geq 0, \beta \geq 0\}$. Then (A, C) is an OBA with C 1-normal and closed that satisfies (1) and (3), but not (2). Moreover, the identity map of A onto $\mathbb{R}^2$ is an order-isomorphism, but not an isometric isomorphism.

Proof: Direct calculation shows that (A, C) is an OBA, the identity has norm 1 and C is 1-normal. Since we can take $c = (2, 2)$, A satisfies (1). If we take $b_n = (1, 1) = c_n$ we see that A satisfies condition (3). Because $(2, 2) \cdot (2, 2) \not\leq (2, 2)$, (2) is not satisfied. The last statement is easy to verify. $\square$

Example 4.32 *Let X be a compact Hausdorff space and A the ordered Banach space $C(X)$ ordered by the cone $C := C^+(X)$. We define the product of any two functions to be zero. Then A is an OBA with C 1-normal and closed, that satisfies (1) and (2), but not (3).*

Proof: Straightforward. $\square$

Chapter 5

The boundary spectrum in OBA's

Following [22] we define the *boundary spectrum* and investigate some properties of this set and its relation with the spectral radius.

Let A be a Banach algebra with unit 1 and let S (or S_A if necessary) be the set of all non-invertible elements of A. Then S is closed. Now we define the boundary spectrum.

Definition 5.1 If A is a Banach algebra with identity and $a \in A$ then the boundary spectrum of a, denoted by $S_\partial(a)$, is defined by

$$S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda - a \in \partial S\}.$$

We also define related to radii r_1 and r_2,

$$r_1(a) = \sup\{|\lambda| : \lambda \in \partial\sigma(a)\},$$

$$r_2(a) = \sup\{|\lambda| : \lambda \in S_\partial(a)\}.$$

Proposition 5.2 *Let A be a Banach algebra and $a \in A$. Then $\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a)$; and therefore $r_1(a) = r_2(a) = r(a)$ and if $\alpha \notin \sigma(a)$, then $d(\alpha, \partial\sigma(a)) = d(\alpha, S_\partial(a)) = d(\alpha, \sigma(a))$.*

Proof: Let $\lambda \in \partial\sigma(a)$ and $\epsilon > 0$. Then there are $\lambda_1 \in B(\lambda, \epsilon) \cap \sigma(a)$ and $\lambda_2 \in B(\lambda, \epsilon) \cap (\mathbb{C} \setminus \sigma(a))$. Let $b_1 = \lambda_1 - a$ and $b_2 = \lambda_2 - a$, then $b_1 \in S$, $b_2 \notin S$ and $b_1, b_2 \in B(\lambda - a, \epsilon)$. Therefore $\lambda - a \in \partial S$, so by definition $\lambda \in S_\partial(a)$. Thus $\partial\sigma(a) \subseteq S_\partial(a)$, and since S is closed, $\partial S \subseteq S$, so that $S_\partial(a) \subseteq \sigma(a)$. The rest is then clear. $\square$

Because $\sigma(a)$ is non-empty, it follows from Proposition 5.2 that $S_\partial(a)$ is non-empty. Since ∂S is closed, $S_\partial(a)$ is closed, so it is a closed subset of the compact set $\sigma(a)$ and therefore it is compact as well.

Proposition 5.3 *Let a be an invertible element of a Banach algebra A. Then $S_\partial(a^{-1}) = (S_\partial(a))^{-1}$.*

Before we can give the proof of this proposition we need the following lemma:

Lemma 5.4 *Let A be a Banach algebra and $a \in \partial S$ and d an invertible element. Then $ad \in \partial S$ and $da \in \partial S$.*

Proof: Let $a \in \partial S$ and d an invertible element, then for all $\epsilon > 0$ there exists a $c_1 \in S \cap B(a, (\epsilon/\|d\|))$ and a $c_2 \in (A \setminus S) \cap B(a, (\epsilon/\|d\|))$. It follows that $c_1 d \in S \cap B(ad, \epsilon)$ and $c_2 d \in (A \setminus S) \cap B(ad, \epsilon)$. Hence $ad \in \partial S$ and similarly $da \in \partial S$. □

Now we give the proof of Proposition 5.3.

Proof: Let $a \in A$ be invertible. If $\lambda \in S_\partial(a^{-1})$, then $\lambda - a^{-1} = \lambda(a - (1/\lambda))a^{-1} \in \partial S$. It follows from Lemma 5.4 that $a - (1/\lambda) \in \partial S$, so that $1/\lambda \in S_\partial(a)$ and thus $S_\partial(a^{-1}) \subseteq (S_\partial(a))^{-1}$. Because a^{-1} is invertible as well, we have $S_\partial(a) \subseteq (S_\partial(a^{-1}))^{-1}$ and therefore $(S_\partial(a))^{-1} \subseteq S_\partial(a^{-1})$. □

Using the boundary spectrum we get a stronger version of Theorem 2.7.

Theorem 5.5 *Let (A, C) be an OBA with C closed and such that the spectral radius in (A, C) is monotone. If $a \in C$, then $r(a) \in S_\partial(a)$.*

Proof: If $a \in C$, then by Theorem 2.8 $r(a) \in \sigma(a)$. Hence $r(a) \in \partial\sigma(a)$ and so $r(a) \in S_\partial(a)$. □

Theorem 5.6 *Let (A, C) be an OBA with C closed and inverse-closed, and such that the spectral radius is monotone. If a is an invertible element of C, then $\delta(a) \in S_\partial(a)$.*

Proof: If $a \in C$ and a is invertible, then $a^{-1} \in C$, since C is inverse-closed. From Theorem 5.5 we have $r(a^{-1}) \in S_\partial(a^{-1})$. Therefore $r(a^{-1}) = 1/\lambda_0$ for some $\lambda_0 \in S_\partial(a)$, by Proposition 5.3, and from Lemma 1.23 we know that $r(a^{-1}) = 1/(\delta(a))$, so $\delta(a) = \lambda_0$. □

In the following result B is a subalgebra of A but not necessarily closed in A.

Theorem 5.7 *Let (A, C) be an OBA and B a Banach algebra with $1 \in B \subseteq A$.*

1. *Suppose that the spectral radius in (A, C) is monotone. If $0 \leq a \leq b$ with $a, b \in B$ and $\partial\sigma(a, B) = \partial\sigma(a, A)$ or $S_\partial(a, B) = S_\partial(a, A)$, then $r(a, B) \leq r(b, B)$.*
2. *Suppose that the spectral radius in $(B, B \cap C)$ is monotone. If $0 \leq a \leq b$ with $a, b \in B$ and $\partial\sigma(b, B) = \partial\sigma(b, A)$ or $S_\partial(b, B) = S_\partial(b, A)$, then $r(a, A) \leq r(b, A)$.*

Proof:

1. Since B is a subalgebra of A we have $\sigma(b, A) \subset \sigma(b, B)$ and therefore $r(b, A) \leq r(b, B)$. From the monotonicity of the spectral radius in (A, C) it follows that $r(a, A) \leq r(b, A)$. By Proposition 5.2, the assumption that either $\partial\sigma(a, B) = \partial\sigma(a, A)$ or $S_\partial(a, B) = S_\partial(a, A)$ give us that $r(a, B) = r(a, A)$. So $r(a, B) = r(a, A) \leq r(b, A) \leq r(b, B)$.
2. The proof is similar to the proof in (1).

□

We note that Theorem 5.7.2 is a stronger version of Proposition 2.5.

Chapter 6

Spectral continuity in OBA's

In this chapter we will turn our attention to the continuity of the spectrum and the spectral radius. To define spectral continuity we first introduce the *Hausdorff distance*. Then we state results about the continuity of the spectrum that are valid for Banach algebras, following [5]. After that we focus on OBA's and using [19], [21] and [23] state several results about the continuity of spectrum and the spectral radius in OBA's.

6.1 Continuity of the spectrum.

Let A be a Banach algebra. An important question about the spectrum function $x \mapsto \sigma(x)$, is under which conditions it is continuous. The spectrum function maps to a compact set in $\mathbb{C}$, so in order to define continuity for this map, we introduce a distance on the set of compact subsets of $\mathbb{C}$, called the *Hausdorff distance* . This distance is defined by

$$\Delta(K_1, K_2) = \max\Big(\sup_{z \in K_2} \mathrm{d}(z, K_1), \sup_{z \in K_1} \mathrm{d}(z, K_2)\Big),$$

for K_1 and K_2 compact subsets of $\mathbb{C}$. Let $r > 0$ and K a compact subset of $\mathbb{C}$, the we denote by $K + r$ the set $\{z : \mathrm{d}(z, K) \leq r\}$. From this we see that $K_1 \subset K_2 + \Delta(K_1, K_2)$ and $K_2 \subset K_1 + \Delta(K_1, K_2)$.

If B is a cone, then $\lambda B \subset B$ for all $\lambda > 0$. Also $\lambda\sigma(x) = \sigma(\lambda x)$ and $\Delta(\lambda K_1, \lambda K_2) = \lambda\Delta(K_1, K_2)$. From these properties it follows that uniform continuity of the spectrum on the cone B is equivalent with the condition that there exists a $C > 0$ such that $\Delta(\sigma(x), \sigma(y)) \leq C\|x - y\|$, for $x, y \in B$. We show the non-trivial implication. If the spectrum is uniform continuous on B, then there exists a $\delta > 0$ such that for all $x, y \in B$ we have $\|x - y\| < \delta \Rightarrow \Delta(\sigma(x), \sigma(y)) < 1$. Take $C = 2/\delta$. Let $x, y \in B$ with $x \neq y$, then there is a λ such that $\lambda\|x - y\| = \|\lambda x - \lambda y\| = \delta/2$. So $\Delta(\sigma(\lambda x), \sigma(\lambda y)) = \lambda\Delta(\sigma(x), \sigma(y)) < 1 = C\lambda\|x - y\|$. Thus $\Delta(\sigma(x), \sigma(y)) < C\|x - y\|$.

We will investigate conditions under which the spectrum is (uniformly) continuous. An important spectral property is the following.

Theorem 6.1 *Let A be a Banach algebra. Suppose that $x, y \in A$ satisfy $xy = yx$. Then $r(x+y) \leq r(x) + r(y)$ and $r(xy) \leq r(x)r(y)$.*

Proof: We have $(xy)^n = x^n y^n$, so

$$r(xy) = \lim_{n\to\infty} \|(xy)^n\|^{1/n} \leq \lim_{n\to\infty} \|x^n\|^{1/n} \lim_{n\to\infty} \|y^n\| = r(x)r(y).$$

Let $\alpha > r(x)$, $\beta > r(y)$ and $a = x/\alpha$, $b = y/\beta$. Then $r(a) < 1$ and $r(b) < 1$ and therefore there exists a integer N such that for all $n \geq N$, $\max(\|a^{2^n}\|, \|b^{2^n}\|) < 1$. Define

$$\gamma_n = \max_{0\leq k\leq 2^n} \|a^k\| \cdot \|b^{2^n-k}\|,$$

then we have

$$\begin{aligned}
\|(x+y)^{2^n}\|^{1/2^n} &= \|\sum_{k=0}^{2^n} \binom{2^n}{k} x^k y^{2^n-k}\|^{1/2^n} \\
&\leq \left(\sum_{k=0}^{2^n} \binom{2^n}{k} \alpha^k \beta^{2^n-k} \|a^k\| \cdot \|b^{2^n-k}\|\right)^{1/2^n} \\
&\leq (\alpha+\beta)\gamma_n^{1/2^n}.
\end{aligned}$$

Because

$$\begin{aligned}
\gamma_{n+1} &= \max_{0\leq k\leq 2^{n+1}} \|a^k\| \cdot \|b^{2^{n+1}-k}\| \\
&= \max\left(\max_{0\leq k\leq 2^n} \|a^k\| \cdot \|b^{2^{n+1}-k}\|, \max_{2^n\leq k\leq 2^{n+1}} \|a^k\| \cdot \|b^{2^{n+1}-k}\|\right) \\
&\leq \gamma_n \max(\|a^{2^n}\|, \|b^{2^n}\|).
\end{aligned}$$

we see that the series γ_n is decreasing for $n \geq N$. So we have $r(x+y) = \lim_{n\to\infty} \|(x+y)^{2^n}\|^{1/2^n} \leq (\alpha+\beta) \limsup_{n\to\infty} \gamma_n^{1/2^n} \leq (\alpha+\beta) \limsup_{n\to\infty} \gamma_N^{1/2^n} = \alpha + \beta$, for arbitrary $\alpha > r(x)$, $\beta > r(y)$. Hence $r(x+y) \leq r(x) + r(y)$. $\square$

If A is a commutative Banach algebra, the spectrum is uniformly continuous.

Theorem 6.2 *Let A be a Banach algebra. Suppose that $x, y \in A$ commute. Then $\sigma(y) \subset \sigma(x) + r(x-y)$ and therefore we have $\Delta(\sigma(x), \sigma(y)) \leq r(x-y) \leq \|x-y\|$. Consequently, if A is commutative then the spectrum function is uniformly continuous on A.*

Proof: Suppose the inclusion is not true. Then there exists an $\alpha \in \sigma(y)$ such that $\mathrm{d}(\alpha, \sigma(x)) > r(x-y)$. Lemma 1.23 now tells us that $r((\alpha-x)^{-1})r(x-y) < 1$. So, by Theorem 6.1, we have $r((\alpha-x)^{-1}(x-y)) < 1$, hence $1+(\alpha-x)^{-1}(x-y)$ is invertible. Therefore $\alpha - y = (\alpha-x)(1+(\alpha-x)^{-1}(x-y))$ is also invertible, which is a contradiction. $\square$

If A is a noncommutative algebra, the spectrum function need not to be continuous. However, the spectrum is always upper semicontinuous.

Theorem 6.3 *Let A be a Banach algebra. Then the spectrum function σ is upper semicontinuous on A. That is, if $x \in A$, then for every open set U containing $\sigma(x)$ there exists $\delta > 0$ such that $\|x-y\| < \delta$ implies $\sigma(y) \subset U$.*

Proof: Suppose there exist sequences (y_n) and (α_n) such that $x = \lim_{n\to\infty} y_n$, $\alpha_n \in \sigma(y_n) \cap (\mathbb{C} \setminus U)$. From Theorem 1.5 we have $|\alpha_n| \leq \|y_n\|$, so (α_n) is a bounded sequence in $\mathbb{C}$ and therefore has a convergent subsequence. So without loss of generality we may assume that (α_n) converges, say to α, and we have $(\alpha_n - y_n) \to (\alpha - x)$ as $n \to \infty$. The set $\mathbb{C} \setminus U$ is closed, so $\alpha \notin U$, thus $\alpha - x$ is invertible. The set of invertible elements is open by Theorem 1.3, so for n large enough $\alpha_n - y_n$ will be invertible, which is a contradiction. $\square$

From this theorem we can deduce that the spectral radius function is upper semicontinuous as well.

Corollary 6.4 *Let A be a Banach algebra. Then the spectral radius function r is upper semicontinuous on A.*

Proof: Let $x \in A$ and $\epsilon > 0$. Let $U := \bigcup_{\alpha\in\sigma(x)} B_\epsilon(\alpha)$. Then U is an open set containing $\sigma(x)$, so from Theorem 6.3 it follows that there exists a $\delta > 0$ such that $\|x - y\| < \delta$ implies $\sigma(y) \subset U$. Therefore $\|x - y\| < \delta$ implies that $\sup_{z\in\sigma(y)} \mathrm{d}(z, \sigma(x)) < \epsilon$, thus $r(y) < r(x) + \epsilon$. $\square$

Now two important results by J.D. Newburgh.

Theorem 6.5 (J.D. Newburgh) *Let A be a Banach algebra and $x \in A$. Suppose that U, V are two disjoint open sets such that $\sigma(x) \subset U \cup V$ and $\sigma(x) \cap U \neq 0$. Then there exists $r > 0$ such that $\|x - y\| < r$ implies $\sigma(y) \cap U \neq 0$.*

Proof: Since the spectrum is upper semicontinuous, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\sigma(y) \subset U \cup V$. Therefore, if the theorem is false, there exists a sequence (y_n) converging to x such that $\sigma(y) \subset V$ for n large enough. Let f be the function on $U \cup V$ defined by 1 on U and 0 on V. Then f is holomorphic on $U \cup V$ and from the definition of the functional calculus we see that $\lim_{n\to\infty} f(y_n) = f(x)$ and $f(y_n) = 0$ for n large enough. So using the spectral mapping theorem we have $0 = \sigma(0) = \sigma(f(x)) = f(\sigma(x))$. But $f(\sigma(x))$ contains 1, which gives a contradiction. $\square$

Definition 6.6 A topological space X is totally disconnected if for every $x \in X$ and every open neighbourhood U of x there is a $V \subset X$ that is both open and closed and such that $x \in V \subseteq U$.

Corollary 6.7 (J.D. Newburgh) *Suppose that the spectrum of a is totally disconnected. Then $x \mapsto \sigma(x)$ is continuous at a.*

Proof: Let $\epsilon > 0$. Since $\sigma(a)$ is totally disconnected it is included in the union U, say $\bigcup_{i=1}^{k} U_k$, of a finite number of disjoint open sets U_i, intersecting $\sigma(a)$ and with diameters smaller then ϵ. By Theorem 6.2 there exists an $r_0 > 0$ such that $\|x - a\| < r_0$ implies that $\sigma(x) \subset U$. Since for all $1 \leq i \leq k$ we have $\sigma(a) \cap U_i \neq \emptyset$, we can apply Theorem 6.5 to $U_i \cup (U \setminus \{U_i\})$. It follows that there exists an $r_i > 0$ such that $\|x - a\| < r_i$ implies $\sigma(x) \cap U_i \neq \emptyset$. So for $r = \min(r_1, \ldots, r_k)$ we have that $\|x - a\| < r$ implies that $\mathrm{dist}(z, \sigma(x)) < \epsilon$ for $z \in \sigma(a)$. So $\|x - a\| < \min(r_0, r)$ implies $\Delta(\sigma(a), \sigma(x)) < \epsilon$. $\square$

This corollary implies in particular that the spectral function is continuous at all elements having finite or countable spectrum.

We state one more theorem about general spectral continuity. It is a result of K. Kuratowski and tells us that even if the spectrum is discontinuous, the set of elements where it is continuous is dense in the algebra.

Theorem 6.8 *Let A be a Banach algebra. Then the set of points of continuity of $x \mapsto \sigma(x)$ is a dense G_δ-subset of A.*

Proof: This is Theorem 3.4.3. in [5]. $\square$

6.2 Continuity of the spectral radius

Now we turn our attention to continuity of the spectral radius in ordered Banach algebras.

From Theorem 2.8 we get the following lemma.

Lemma 6.9 *Let (A, C) be an OBA with C closed and normal. If $x \in C$ and $\alpha \in \mathbb{R}^+$, then $r(x + \alpha) = r(x) + \alpha$.*

Now we define the following set.

Definition 6.10 Let (A, C) be an OBA, define for each $x \in C$ the set $A(x)$ by,

$$\begin{aligned} A(x) &= \{y \in A : x \leq y, (xy \leq yx \text{ or } yx \leq xy), \\ &\quad \text{and } \mathrm{d}(r(y), \sigma(x)) \geq \mathrm{d}(\alpha, \sigma(x)) \text{ for all } \alpha \in \sigma(y)\} \end{aligned}$$

From this definition we easily see $x \in A(x)$, $A(x) \subset C$ and $A(0) = C$. Lemma 6.9 tells us that if C is closed and normal, then $A(\alpha) = C + \alpha$ for all $\alpha \in \mathbb{R}^+$.

In Theorem 6.2 we saw that for all y in the commutant $\{x\}'$ of x we have $\sigma(y) \subset \sigma(x) + r(x - y)$. We are going to prove a theorem which shows that this inclusion will hold for positive elements x of an OBA, if y is an element of $A(x)$ rather than of the commutant of x.

First we need the following lemma, which was proved in the proof of Theorem 6.2.

Lemma 6.11 *Let A be a Banach algebra, $x, y \in A$ and $\alpha \in \mathbb{C}$. If $\alpha - x$ is invertible and $r((\alpha - x)^{-1}(x - y)) < 1$, then $\alpha - y$ is invertible.*

Theorem 6.12 *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then $\sigma(y) \subset \sigma(x) + r(x - y)$ for all $y \in A(x)$.*

Proof: Let $y \in A(x)$. Then $0 \leq x \leq y$, so that $r(x) \leq r(y)$ by Theorem 2.2. If $r(x) = r(y)$, then $\mathrm{d}(r(y), \sigma(x)) = 0$ by Theorem 2.8, and by the last condition in the definition of $A(x)$, we see from the assumption that $y \in A(x)$ that $\mathrm{d}(\alpha, \sigma(x)) = 0$ for all $\alpha \in \sigma(y)$. So $\mathrm{d}(\alpha, \sigma(x)) \leq r(x - y)$ for all $\alpha \in \sigma(y)$, and we have $\sigma(y) \subset \sigma(x) + r(x - y)$.

So suppose that $r(x) < r(y)$ and suppose that there exists an $\alpha \in \sigma(y)$ such that $\mathrm{d}(\alpha, \sigma(x)) > r(x - y)$. By Theorem 2.8 we know that $r(y) \in \sigma(y)$. By the assumption that $y \in A(x)$, we see from the definition of $A(x)$ that

$d(r(y), \sigma(x)) \geq d(\alpha, \sigma(x)) > r(x-y)$ and therefore we may take such an $\alpha \in \mathbb{R}^+$ with $\alpha > r(x)$ for example, $\alpha = r(y)$. Now it follows from Lemma 1.23 that

$$r((\alpha - x)^{-1})r(x - y) < 1. \tag{6.1}$$

It follows from Theorem 3.12 that $(\alpha - x)^{-1} \in C$ and because $x \leq y$, we have $(y - x) \in C$. If $xy \leq yx$, then $(y - x)(\alpha - x) \leq (\alpha - x)(y - x)$, so $(\alpha - x)^{-1}(y - x) \leq (y - x)(\alpha - x)^{-1}$ by Lemma 1.48. From Proposition 2.4 it now follows that $r((\alpha - x)^{-1}(y - x)) \leq r((\alpha - x)^{-1})r(y - x)$. In the same way we get this result in the case $yx \leq xy$.

This result together with equation (6.1) gives us $r((\alpha - x)^{-1}(y - x)) < 1$ and from Lemma 6.11 it follows that $\alpha \notin \sigma(y)$, which is a contradiction. Thus $d(\alpha, \sigma(x)) \leq r(x - y)$ for all $\alpha \in \sigma(y)$ and therefore we have $\sigma(y) \subset \sigma(x) + r(x - y)$. □

From this theorem it does not follow that the restriction of the spectrum function to the set $A(x)$ is continuous in x, because we only prove that $\sup_{y \in \sigma(y)} d(y, \sigma(x)) \leq r(x - y)$. Because x and y are not interchangeable in the theorem, $\sup_{x \in \sigma(x)} d(x, \sigma(y)) \leq r(x - y)$ does not have to be true. We can however use this theorem to prove that the spectral radius function restricted to $A(x)$ is continuous in x.

Theorem 6.13 *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then the spectral radius restricted to $A(x)$ is continuous in x. In fact, if $y \in A(x)$, then $|r(y) - r(x)| \leq r(y - x) \leq \|y - x\|$.*

Proof: Let $y \in A(x)$. If $\lambda \in \sigma(y)$, then $d(\lambda, \sigma(x)) = |\lambda - \mu_\lambda|$ for some $\mu_\lambda \in \sigma(x)$ because $\sigma(x)$ is closed. So it follows from Theorem 6.12 that $|\lambda| \leq |\lambda - \mu_\lambda| + |\mu_\lambda| \leq r(x - y) + r(x)$ for all $\lambda \in \sigma(y)$. Thus $r(y) \leq r(x) + r(x - y)$ for all $y \in A(x)$.

Let $y \in A(x)$, then $x \leq y$ and thus $r(x) \leq r(y)$ by Theorem 2.2. Therefore $|r(y) - r(x)| = r(y) - r(x) \leq r(y - x) \leq \|y - x\|$. □

We can formulate a stronger version of Theorem 6.13. In order to do this we first need the following theorem.

Theorem 6.14 *Let (A, C) be an OBA with C normal, and let $x, y \in C$ be such that either $xy \leq yx$ or $yx \leq xy$. Then $r(x + y) \leq r(x) + r(y)$ and $r(xy) \leq r(x)r(y)$.*

Proof: Without loss of generality assume that $yx \leq xy$. By Lemma 1.49,

$$0 \leq (x + y)^{2^n} \leq \sum_{k=0}^{2^n} \binom{2^n}{k} x^{2^n - k} y^k.$$

Since C is normal, there exists an $\alpha > 0$ such that

$$\|(x + y)^{2^n}\| \leq \alpha \|\sum_{k=0}^{2^n} \binom{2^n}{k} x^{2^n - k} y^k\|$$

and hence

$$\|(x + y)^{2^n}\|^{1/2^n} \leq \alpha^{1/2^n} \|\sum_{k=0}^{2^n} \binom{2^n}{k} x^{2^n - k} y^k\|^{1/2^n}$$

Now we can follow the proof of Theorem 6.1, which leads to the first statement. The last statement was proved in Proposition 2.4. $\square$

With the use of this theorem, we can prove the stronger version of Theorem 6.13.

Theorem 6.15 *Let (A, C) be an OBA with C normal, and let $x \in C$. Then the spectral radius is continuous at x, if it is restricted to the set*

$$A'(x) := \{y \in A : x \leq y, \text{ and } (xy \leq yx \text{ or } yx \leq xy)\}.$$

In fact, if $y \in A'(x)$, then $|r(y) - r(x)| \leq \|y - x\|$.

Proof: Let $y \in A'(x)$. Then $(y-x) \in C$ because $x \leq y$ and the condition $xy \leq yx$ or $yx \leq xy$ implies respectively that $x(y-x) \leq (y-x)x$ or $(y-x)x \leq x(y-x)$. So it follows from Theorem 6.14 that $r(y) = r(x + (y - x)) \leq r(x) + r(y - x)$ and the result follows as in 6.13. $\square$

With $C(0, r(x))$ we will denote the circle centered at 0 with radius $r(x)$.

Theorem 6.16 *Let A be a Banach algebra and let $x \in A$ be such that $\sigma(x) \subset C(0, r(x))$. Then the spectral radius is continuous at x.*

Proof: Let $\epsilon > 0$, and $G_\epsilon = \{\lambda \in \mathbb{C} : r(x) - \epsilon < |\lambda| < r(x) + \epsilon\}$. Then $\sigma(x) \subset G_\epsilon$. If $x_n \to x$, then by the upper semicontinuity of the spectrum there exists an $N \in \mathbb{N}$ such that $\sigma(x_n) \subset G_\epsilon$ for all $n \geq N$. Since $r(x_n) = |\lambda_n|$ for some $\lambda_n \in \sigma(x_n)$, it follows that $r(x)-\epsilon < r(x_n) < r(x)+\epsilon$, so $|r(x) - r(x_n)| < \epsilon$ for all $n \geq N$. $\square$

Now we define for each $x \in C$ the set

$$\begin{aligned} B(x) \quad = \quad & \{y \in A : x \leq y, (xy \leq yx \text{ or } yx \leq xy), \\ & \quad \text{and } (\alpha - x)^{-1} \in C \text{ for all } \alpha \in \sigma(y) \setminus \sigma(x)\}. \end{aligned}$$

Then $x \in B(x)$, $B(x) \subset C$ and $B(0) = \{y \in C : \alpha^{-1} \in C \text{ for all } \alpha \in \sigma(y) \setminus \{0\}\}$, thus if C is proper $B(0) = \{y \in C : \sigma(y) \subset \mathbb{R}_{\geq 0}\}$. We have the following theorem.

Theorem 6.17 *Let (A, C) be an OBA with C normal, and let $x \in C$. Then $\sigma(y) \subset \sigma(x) + r(x - y)$ for all $y \in B(x)$.*

Proof: Let $y \in B(x)$. Suppose there exists an $\alpha \in \sigma(y)$ such that $\mathrm{d}(\alpha, \sigma(x)) > r(x - y)$. From Lemma 1.23 it follows that $r((\alpha - x)^{-1}) = 1/\mathrm{d}(\alpha, \sigma(x)) < 1/r(x - y)$. So we have

$$r((\alpha - x)^{-1})r(x - y) < 1. \tag{6.2}$$

If $xy \leq yx$, then $(y - x)(\alpha - x) \leq (\alpha - x)(y - x)$, so that $(\alpha - x)^{-1}(y - x) \leq (y - x)(\alpha - x)^{-1}$, by Lemma 1.48. Since $y \in B(x)$ we have $y - x \in C$ and $(\alpha - x)^{-1} \in C$, so it follows from Proposition 2.4 that $r((\alpha - x)^{-1}(x - y)) \leq r((\alpha - x)^{-1})r(x - y)$. In the same way we get this result in the case $yx \leq xy$.

This result together with equation (6.2) gives us $r((\alpha - x)^{-1}(y - x)) < 1$ and from Lemma 6.11 it follows that $\alpha \notin \sigma(y)$, which is a contradiction.

Thus $\mathrm{d}(\alpha, \sigma(x)) \leq r(x-y)$ for all $\alpha \in \sigma(y)$ and therefore we have $\sigma(y) \subset \sigma(x) + r(x-y)$. $\square$

From this theorem, as with $A(x)$, we get the continuity of the spectral radius in x, restricted to $B(x)$.

Corollary 6.18 *Let (A, C) be an OBA with C normal, and let $x \in C$. Then the spectral radius restricted to $B(x)$ is continuous in x. In fact, if $y \in B(x)$, then $|r(y) - r(x)| \leq r(y-x) \leq \|y-x\|$.*

The inequality $|r(y) - r(x)| \leq r(y-x) \leq \|y-x\|$ holds for elements commuting with x. Now we give some examples to show that the sets $B(x)$ and $A(x)$ contain elements which do not commute with x.

Example 6.19 *Let A be the OBA as in Example 1.53, then there exists $x \in C$ such that that the sets $A(x)$ and $B(x)$ contain elements which do not commute with x.*

Proof: Let

$$x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \cdots \right)$$

Then $x \in C$ and $\sigma(x) = \{0, 1\}$. Let

$$y = \left(\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \cdots \right)$$

Then $x \leq y$ and $\sigma(y) = \{1, 2\}$. Since

$$xy = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \cdots \right), \ yx = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \cdots \right),$$

we have $yx \leq xy$. The only element of $\sigma(y) \setminus \sigma(x)$ is 2, and

$$(2-x)^{-1} = \left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \cdots \right) \in C$$

So $y \in B(x)$. Since $\mathrm{d}(r(y), \sigma(x)) = 1$ and $\{\delta(\alpha, \sigma(x)) : \alpha \in \sigma(y)\} = \{0, 1\}$, it follows that $y \in A(x)$. $\square$

Now we look at the boundary spectrum end define an analogue $F(x)$ of the set $A(x)$.

Definition 6.20 Let (A, C) be an OBA. Define for each $x \in C$ the set $F(x)$ by,

$$\begin{aligned} F(x) \quad = \quad & \{y \in A : x \leq y, xy \leq yx \text{ or } yx \leq xy, \\ & \text{and } \mathrm{d}(r(y), S_\partial(x)) \geq \mathrm{d}(\alpha, S_\partial(x)) \text{ for all } \alpha \in S_\partial(y)\} \end{aligned}$$

From this definition we see $x \in F(x)$, $F(x) \subset C$ and $F(0) = C$.

Analogous to Theorem 6.12 we have the following theorem.

Theorem 6.21 *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then $S_\partial(y) \subset S_\partial(x) + r(x-y)$ for all $y \in F(x)$.*

Proof: The proof follows the same line as the proof of Theorem 6.12. Let $y \in F(x)$. Then $0 \leq x \leq y$, so $r(x) \leq r(y)$ by Theorem 2.2. If $r(x) = r(y)$, then $\mathrm{d}(r(y), S_\partial(x)) = 0$ by Theorem 5.5. From the last condition in the definition of $F(x)$, we see from the assumption that $y \in F(x)$ that $\mathrm{d}(\alpha, S_\partial(x)) = 0$ for all $y \in S_\partial(y)$. Hence $S_\partial(y) \subset S_\partial(x) \subset S_\partial(x) + r(x-y)$.

So suppose that $r(x) < r(y)$ and suppose that there exists an $\alpha \in S_\partial(y)$ such that $\mathrm{d}(\alpha, S_\partial(x)) > r(x-y)$. By Theorem 5.5 we know that $r(y) \in S_\partial(y)$. By the assumption that $y \in F(x)$, we see from the definition of $F(x)$ that $\mathrm{d}(r(y), S_\partial(x)) \geq \mathrm{d}(\alpha, S_\partial(x)) > r(x-y)$ and therefore we may take such an $\alpha \in \mathbb{R}^+$ with $\alpha > r(x)$ for example, $\alpha = r(y)$. Since $\alpha \notin \sigma(x)$, it follows from Proposition 5.2 that $\mathrm{d}(\alpha, S_\partial(x)) = \mathrm{d}(\alpha, \sigma(x))$, so that $\mathrm{d}(\alpha, S_\partial(x)) = 1/(r((\alpha - x)^{-1}))$ by Lemma 1.23. We thus have

$$r((\alpha - x)^{-1})r(x-y) < 1. \tag{6.3}$$

for some $\alpha > r(x)$.

It follows from Theorem 3.12 that $(\alpha - x)^{-1} \in C$ and because $x \leq y$, we have $(y - x) \in C$. If $xy \leq yx$, then $(x-y)(\alpha - x) \leq (\alpha - x)(y-x)$, so $(\alpha - x)^{-1}(y-x) \leq (y-x)(\alpha-x)^{-1}$ by Lemma 1.48. From Theorem 2.4 it now follows that $r((\alpha - x)^{-1}(y-x)) \leq r((\alpha-x)^{-1})r(y-x)$. In the same way we get this result in the case $yx \leq xy$.

This result together with equation (6.3) gives us $r((\alpha - x)^{-1}(y-x)) < 1$ and from Lemma 6.11 it follows that $\alpha \notin \sigma(y)$. Hence $\alpha \notin S_\partial(y)$, which is a contradiction. Thus $\mathrm{d}(\alpha, S_\partial(x)) \leq r(x-y)$ for all $\alpha \in S_\partial(y)$ and therefore we have $S_\partial(y) \subset S_\partial(x) + r(x-y)$. $\square$

From this theorem we can, as in Theorem 6.13 deduce that the spectral radius restricted to $F(x)$ is continuous in x.

Corollary 6.22 *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then the spectral radius restricted to $F(x)$ is continuous in x. In fact, if $y \in F(x)$, then $|r(y) - r(x)| \leq r(y-x) \leq \|y - x\|$.*

We will now set out to prove that, if $a \in C$ and $S_\partial(a) \cap \mathbb{R}^+ = \{r(a)\}$ the spectral radius restricted to C is continuous in a. We need some preparation.

Definition 6.23 Let A be a Banach algebra with identity and $a \in A$ then we define the set $T(a)$ by,

$$T(a) = \{\lambda \in \mathbb{C} : |\lambda| - a \in \partial S\}$$

and if $T(a) \neq \emptyset$,

$$\gamma(a) = \sup\{|\lambda| : \lambda \in T(a)\}$$

Now there are some easy facts. We have $\lambda \in T(a)$ if and only if $|\lambda| \in T(a)$. Furthermore,

$$T(a) = \{\lambda \in \mathbb{C} : |\lambda| \in S_\partial(a)\} = \{\lambda \in \mathbb{C} : |\lambda| \in S_\partial(a) \cap \mathbb{R}^+\}$$

Hence $T(a) \subset \overline{B}(0, r(a))$ and $T(a)$ is closed, thus $T(a)$ is compact. If $\lambda_0 \in \mathbb{R}^+$, then $S_\partial(a) \cap \mathbb{R}^+ = \{\lambda_0\}$ if and only if $T(a) = C(0, \lambda_0)$. Also $\gamma(a) \in T(a)$ for all $a \in A$. We have the following lemma:

Lemma 6.24 *Let (A, C) be an OBA with C closed and normal. If $a \in C$, then $\gamma(a) = r(a)$ and $r(a) \in T(a)$.*

Proof: Since $T(a) \subset \overline{B}(0, r(a))$ for all $a \in A$ we have (if $T(a) \neq \emptyset$) $\gamma(a) \leq r(a)$. If $a \in C$, then by Theorem 2.2 and Theorem 5.5, $r(a) \in S_\partial(a)$. Since $r(a) \in R^+$, it follows that $r(a) \in T(a)$, so that $T(a) \neq \emptyset$ and $r(a) \leq \gamma(a)$. □

We see from this lemma that for $a \in C$ the set $T(a)$ is not empty. However for $a \notin C$ it can occur that $T(a)$ is empty, and that the properties $\gamma(a) = r(a)$ and $r(a) \in T(a)$ do not hold. This is illustrated with the following example:

Example 6.25 *Let A be the OBA of Example 1.51. Then there exist*

1. *an $a_1 \notin C$ such that $T(a_1) = \emptyset$, and*
2. *an $a_2 \notin C$ such that $T(a_2) \neq \emptyset$, but $r(a_2) \notin T(a_2)$ and $\gamma(a_2) \neq r(a_2)$.*

Proof: We have $\partial S = S$ and hence, if $\begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & \lambda_4 \end{pmatrix} \in A$, then $S_\partial(a) = \{\lambda_1, \lambda_4\} = \sigma(a)$ and $T(a) = \{\lambda \in \mathbb{C} : |\lambda| = \lambda_1 \text{ or } |\lambda| = \lambda_4\}$.

1. Let $a_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Then $a_1 \notin C$ and $T(a_1) = \emptyset$.
2. Let $a_1 = \begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix}$. Then $a_2 \notin C$, $r(a_2) = 3$ and $T(a_2) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, so that $T(a_2) \neq \emptyset$ and $\gamma(a_2) = 1$. It follows that $r(a_2) \notin T(a_2)$ and $\gamma(a_2) \neq r(a_2)$.

□

If A is any complex Banach algebra with unit 1, then $T(-1) = \{\lambda \in \mathbb{C} : |\lambda| \in \{-1\}\} = \emptyset$. So if (A, C) is an OBA with C closed and normal, then there exists an $a \notin C$ such that $T(a) = \emptyset$.

We now prove that the map $a \mapsto T(a)$ is upper semicontinuous.

Theorem 6.26 *Let A be a Banach algebra. The function $a \mapsto T(a)$ from A into $K(\mathbb{C})$, the compact sets of $\mathbb{C}$, is upper semicontinuous on A.*

Proof: Suppose the map is not upper semicontinuous, then there exists an $a \in A$, an open set U containing $T(a)$ and for each $n \in \mathbb{N}$ an $a_n \in A$ such that $a_n \to a$ and $n \to \infty$, but $T(a_n) \notin U$, say $\lambda_n \in T(a_n) \cap \mathbb{C} \setminus U$. Since $\lambda_n \in T(a_n)$, we have $|\lambda_n| \leq r(a_n) \leq \|a_n\|$ and because a_n is a convergent sequence it is bounded. Hence (λ_n) is bounded and it has a convergent subsequence, say $\lambda_{n_k} \to \lambda$ as $k \to \infty$.

Since U is open and $\lambda_n \notin U$, it follows that $\lambda \notin U$. So $\lambda \notin T(a)$ and thus $|\lambda| - a \notin \partial S$. It follows that for some $\epsilon > 0$,

$$\text{either } B(|\lambda| - a, \epsilon) \subseteq S \text{ or } B(|\lambda| - a, \epsilon) \subseteq S^c. \tag{6.4}$$

Since $\lambda_{n_k} \to \lambda$ and $a_{n_k} \to a$ as $k \to \infty$, there is an $N \in \mathbb{N}$ such that $|\lambda_N| - a_N \in B(|\lambda| - a, \epsilon)$. Let $\rho = \epsilon - \|(|\lambda_N| - a_N) - (|\lambda| - a)\|$. Then $\rho > 0$ and $B(|\lambda_N| - a_N, \rho) \subseteq B(|\lambda| - a, \epsilon)$.

Since $\lambda_N \in T(a_N)$, we have that $|\lambda_N| - a_N \in \partial S$ and therefore $B(|\lambda_N| - a_N, \rho)$ contains a point of S as well as a point of S^c. But this means that $B(|\lambda| - a, \epsilon)$ contains a point of both S and S^c, which is a contradiction with equation (6.4). So the map is upper semicontinuous. $\square$

Now we can use the upper semicontinuity of the map $a \mapsto T(a)$ to prove the following theorem.

Theorem 6.27 *Let (A, C) be an OBA with C closed and normal and let $a \in C$ be such that $S_\partial(a) \cap \mathbb{R}^+ = \{r(a)\}$. Then the spectral radius restricted to C is continuous in a.*

Proof: Let (a_n) be a sequence in C such that $a_n \to a$ as $n \to \infty$ and let (λ_n) be a sequence in $T(a)$ such that $\mathrm{d}(r(a_n), T(a)) = |r(a_n) - \lambda_n|$. Now let $\epsilon > 0$ and $U = \{\lambda \in \mathbb{C} : \mathrm{d}(\lambda, T(a)) < \epsilon\}$. Then U is open and $T(a) \subseteq U$. The map $x \mapsto T(x)$ is upper semicontinuous on A by Theorem 6.26 and $a_n \to a$, therefore there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $T(a_n) \subseteq U$. Lemma 6.24 tells us that $r(a_n) \in T(a_n) \subseteq U$, so $|r(a_n) - \lambda_n| < \epsilon$ for all $n > N$, so $|r(a_n) - |\lambda_n|| < \epsilon$ for all $n > N$. Since $\lambda_n \in T(a)$, it follows that $|\lambda_n| \in S_\partial(a) \cap \mathbb{R}^+$, so that by the assumption that $S_\partial(a) \cap \mathbb{R}^+ = \{r(a)\}$ we have $|\lambda_n| = r(a)$, for all $n \in \mathbb{N}$. Therefore $|r(a_n) - r(a)| < \epsilon$ for all $n > N$. $\square$

6.3 Convergence properties

In this section we use general OBA theory and functional analysis to come to several convergence results for specific points in the spectrum.

Theorem 6.28 *Let A be a Banach algebra. Suppose that (a_n) is a sequence in A such that $a_n \to a \in A$. If $\mathrm{psp}(a)$ contains at least one point that is isolated in $\sigma(a)$ then the following properties hold:*

1. *$r(a_n) \to r(a)$ as $n \to \infty$.*
2. *If (α_n) is a sequence such that $\alpha_n \in \mathrm{psp}(a)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ then $\alpha \in \mathrm{psp}(a)$.*

Proof:

(1) Let $\mu \in \mathrm{psp}(a)$ be isolated in σ. Let $B(\mu, r)$ be an open ball such that $B(\mu, r) \cap \sigma(a) = \mu$, and let V be an open set such that $B(\mu, r)$ and V are disjoint and $\sigma(a) \setminus \mu \subset V$. Let $0 \leq \epsilon \leq r$. Then $\sigma(a) \subset B(0, r(a) + \epsilon)$. From the upper semicontinuity of the spectral radius (Corollary 6.4) and the fact that $a_n \to a$, it follows that there exists an $N_{\epsilon,1} \in \mathbb{N}$ such that $r(a_n) \leq r(a) + \epsilon$, for all $n \geq N_{\epsilon,1}$. We have $\sigma(a) \subset B(\mu, \epsilon) \cup V$ and $\sigma(a) \cap B(\mu, \epsilon) \neq \emptyset$, so according to Newburgh's theorem (Theorem 6.5) there exists an $N_{\epsilon,2} \in \mathbb{N}$ such that $\sigma(a_n) \cap B(\mu, \epsilon) \neq \emptyset$, say $\alpha_n \in \sigma(a_n)$ and $|\alpha_n - \mu_1| < \epsilon$, for all $n \geq N_{\epsilon,2}$. Then $r(a_n) \geq |\alpha_n| > r(a) - \epsilon$ for all $n \geq N_{\epsilon,2}$. Let $N := \max\{N_{\epsilon,1}, N_{\epsilon,2}\}$. Then it follows that, if $n \geq N$, then $r(a) - \epsilon < r(a_n) < r(a) + \epsilon$.

(2) This follows from Theorem 1.7 and (1).

$\square$

Theorem 6.29 *Let (A, C) be an OBA with C closed and suppose the spectral radius in (A, C) is monotone. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$. If $r(a)$ is a Riesz point of $\sigma(a)$, then there is a natural number N such that, for all $n \geq N$, $r(a_n)$ is a Riesz point of $\sigma(a_n)$.*

Proof: By the upper semicontinuity of the spectral radius $\limsup r(\overline{a_n}) \leq r(\overline{a})$, and by Lemma 2.12.1 we have $r(\overline{a}) < r(a)$. Let $\lambda \in \mathbb{R}$ be such that $r(\overline{a}) < \lambda < r(a)$, then there exists an $N_1 \in \mathbb{N}$ such that, for $n \geq N_1$, we have $r(\overline{a}_n) < \lambda$. Theorem 6.28 tells us that there exists an $N_2 \in \mathbb{N}$ such that, for $n \geq N_2$, we have $r(a_n) \geq \lambda$. So if $n \geq \max\{N_1, N_2\}$, then $r(\overline{a_n}) < r(a_n)$. By Lemma 2.12, $r(a_n)$ is a Riesz point of $\sigma(a_n)$. $\square$

The above theorem can be extended to any sequence (α_n) where each α_n is an element of the boundary of the unbounded connected component of the resolvent set of a_n, and (α_n) converges to an element α in the peripheral spectrum of a. This result can be proved without assuming that the spectral radius in (A, C) is monotone:

Theorem 6.30 *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in C$, $a_n \in A$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$, and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha \in \operatorname{psp}(a)$, $\alpha_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then there is a natural number N such that, for all $n \geq N$, α_n is a Riesz point of $\sigma(a_n)$.*

Proof: Suppose that there exists a subsequence (α_{n_k}) of (α_n) such that α_{n_k} is an element of the connected hull of $\sigma(\overline{a_{n_k}})$, so $\alpha_{n_k} \in \sigma(\overline{a_{n_k}})^\wedge$ for all $k \in \mathbb{N}$. Then $|\alpha_{n_k}| \leq r(\overline{a_{n_k}})$ for all $k \in \mathbb{N}$ and since $\alpha_n \to \alpha \in \operatorname{psp}(a)$ we have $|\alpha_{n_k}| \to r(a)$. Therefore because of the upper semicontinuity of the spectral radius, $r(a) \leq \limsup r(\overline{a_{n_k}}) \leq r(\overline{a})$ and we get $r(a) = r(\overline{a})$. By Lemma 2.12.1 this is in contradiction with the fact that $r(a)$ is a Riesz point of $\sigma(a)$, thus there exists an $N \in \mathbb{N}$ such that $\alpha_n \notin \sigma(\overline{a_n})^\wedge$ for all $n \geq N$. Because $\alpha_n \in \partial_\infty \sigma(a_n)$ we also have $\alpha_n \in \sigma(a_n)$. Now it follows from Lemma 2.13 that for $n \geq N$, α_n is a Riesz point of $\sigma(a_n)$. $\square$

Now we discuss some general Banach algebra theory as preparation for OBA convergence theorems for coefficients in Laurent series of the resolvent.

The following theorem is Theorem 5.1 in [19]. This theorem appears to be not entirely correct, which we will demonstrate with an example. We do state the theorem and the proof that was given below, and point out what mistake seems to have been made.

Theorem 6.31 *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is an isolated point of $\sigma(a_n)$, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is an isolated point in $\sigma(a)$. Let $r_n := d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\})$, for all $n \in \mathbb{N}$ such that $\sigma(a_n) \setminus \{\alpha_n\} \neq \emptyset$. If $r_n \to s$, then $s \neq 0$.*

Proof: For each $n \in \mathbb{N}$, the distance r_n is the largest number such that $B(\alpha_n, r_n) \cap \sigma(a_n) = \{\alpha_n\}$. Hence for each $n \in \mathbb{N}$ and every $m \in \mathbb{N}$ there exists a $\lambda_{n,m} \in B(\alpha_n, r_n + \frac{1}{m}) \cap \sigma(a_n)$ such that $\lambda_{n,m} \neq \alpha_n$ and $\lambda_{n,m} \notin B(\alpha_n, r_n)$. Because $a_n \to a$, the sequence $(\lambda_{n,m}) = (\lambda_{1,m}, \lambda_{2,m}, \dots)$ is bounded and hence has a convergent subsequence which we again denote by $(\lambda_{n,m})$. For each m denote the limit of the corresponding convergent subsequence by λ_m. Since $\lambda_{n,m} \in \sigma(a_n)$ for all m and n in $\mathbb{N}$, it follows from Lemma 1.7.1 that $\lambda_m \in \sigma(a)$ for all $m \in \mathbb{N}$. We have the inequality $r_n \leq |\lambda_{n,m} - \alpha_n| < r_n + \frac{1}{m}$, where $\lambda_{n,m}$ is an element of the convergent subsequence $(\lambda_{n,m})$. Letting $n \to \infty$, we see that $s \leq |\lambda_m - \alpha| \leq s + \frac{1}{m}$ for all $m \in \mathbb{N}$. So if $s = 0$, then $\lambda_m \to \alpha$ as $m \to \infty$ and therefore α is an accumulation point of $\sigma(a)$. But this is a contradiction with the assumption that α is an isolated point in $\sigma(a)$, so $s \neq 0$. $\square$

The mistake that seems to have been made is the following. The assumption is made that, if $\lambda_m \to \alpha$ as $m \to \infty$ then α is an accumulation point of $\sigma(a)$. However, it seems to be possible that λ_m can be the same for all m, in which case α is not an accumulation point.

We have the following example to show what can go wrong.

Example 6.32 *Let* $A = M_2(\mathbb{C})$, $\alpha = 1$, $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_n = 1 + \frac{1}{n}$, *and* $a_n = \begin{pmatrix} 1 + \frac{1}{n} & 0 \\ 0 & 1 + \frac{1}{n^2} \end{pmatrix}$ *with* $n \in \mathbb{N}$. *Then* α_n *is a pole of the resolvent of* a_n, α *is a pole of the resolvent of* a, $a_n \to a$ *and* $\alpha_n \to 1$ *as* $n \to \infty$. *But* $d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\}) = \dfrac{1}{n} - \dfrac{1}{n^2} \to 0$.

The theorems that follow make use of the result of Theorem 6.31. Since this theorem might not be entirely correct, we added the result as an extra condition in the theorems.

Theorem 6.33 *Let* A *be a Banach algebra and* (a_n) *a sequence in* A *such that* $a_n \to a \in A$. *Suppose that* (α_n) *is a sequence in* $\mathbb{C}$ *such that, for each* $n \in \mathbb{N}$, α_n *is an isolated point of* $\sigma(a_n)$, *and* $\alpha_n \to \alpha \in \mathbb{C}$ *where* α *is an isolated point of* $\sigma(a)$. *Suppose* $\inf_{n \in N} d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\}) > 0$. *If*

$$(\lambda - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j b_j$$

and

$$(\lambda - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_n)^j b_{n,j}$$

are the Laurent series of the resolvents of a *and* a_n, *then* $b_{n,j} \to b_j$ *as* $n \to \infty$, *for all* $j \in \mathbb{Z}$.

Proof: Let $r_n := d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\})$ for all $n \in \mathbb{N}$ such that $\sigma(a_n) \setminus \{\alpha_n\} \neq \emptyset$. Let $r := d(\alpha, \sigma(a) \setminus \{\alpha\})$ if $\sigma(a) \setminus \{\alpha\} \neq \emptyset$ and $r = 1$ if $\sigma(a) \setminus \{\alpha\} = \emptyset$. By assumption, $\inf_{n \in \mathbb{N}} r_n = K_1 > 0$. Define the curves $\Gamma, \Gamma_n : [0, 2\pi] \to \mathbb{C}$ by $\Gamma_n(t) = \alpha_n + Ke^{it}$ and $\Gamma(t) = \alpha + Ke^{it}$ for a fixed $K > 0$ with $K < K_1$. Now Lemma 3.1 gives us

$$\begin{aligned} b_{n,j} &= \frac{1}{2\pi}\int_{\Gamma_n}\frac{(z-a_n)^{-1}}{(z-\alpha_n)^{j+1}}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} g_{n,j}(t)dt \end{aligned}$$

for all $n \in \mathbb{N}$ and

$$\begin{aligned} b_j &= \frac{1}{2\pi}\int_{\Gamma}\frac{(z-a)^{-1}}{(z-\alpha)^{j+1}}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} g_j(t)dt \end{aligned}$$

where $g_{n,j}(t) = (\alpha_n+Ke^{it}-a_n)^{-1}i(Ke^{it})^{-j}$ and $g_j(t) = (\alpha+Ke^{it}-a)^{-1}i(Ke^{it})^{-j}$ are continuous on $[0, 2\pi]$.

Let $f_n(t) = \Gamma_n(t) - a_n = \alpha_n + Ke^{it} - a_n$ and $f(t) = \Gamma(t) - a = \alpha + Ke^{it} - a$ for all $n \in \mathbb{N}$ and for all $t \in [0, 2\pi]$. Then (f_n) converges uniformly to f on $[0, 2\pi]$. Denote with Γ^* and Γ_n^* the ranges of Γ and Γ_n. We have $\Gamma_n^* \subset \rho(a_n)$ for all $n \in \mathbb{N}$ and $\Gamma^* \subset \rho(a)$. Let $B := \cup_{n\in\mathbb{N}}(\Gamma_n^* - a_n) \cup (\Gamma^* - a)$, then $f_n(t), f(t) \in B$ for all $n \in N$ and for all $t \in [0, 2\pi]$. It is not hard to see that, because Ke^{it} is compact and $(\alpha_n - a_n) \to (\alpha - a)$, B is compact. The set B is also contained in the subset of invertible elements of A, so the function $x \mapsto x^{-1}$ is analytic on the compact set B, thus is uniformly continuous on B. Therefore (f_n^{-1}) converges uniformly to f^{-1} on $[0, 2\pi]$. Since $|e^{-ijt}| = 1$ for all $t \in [0, 2\pi]$ and for all $j \in \mathbb{Z}$, it follows that $(g_{n,j})$ converges to g_j uniformly on $[0, 2\pi]$ for each $j \in \mathbb{Z}$, which yields the result. □

From the definition of the spectral projection we know that $b_{n,-1} = p(a_n, \alpha_n)$ and $b_{-1} = p(a, \alpha)$. This leads us immediately to the following corollary.

Corollary 6.34 *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is an isolated point of $\sigma(a_n)$, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is an isolated point of $\sigma(a)$. Suppose $\inf_{n\in N} d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\}) > 0$. Then $p(a_n, \alpha_n) \to p(a, \alpha)$ as $n \to \infty$.*

Corollary 6.35 *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is a pole of the resolvent of a_n of order k_n, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is a pole of the resolvent of a of order k. Suppose $\inf_{n\in N} d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\}) > 0$. Let the Laurent series of the resolvents of a and a_n be as in Theorem 6.33 and $u := b_{-k}$, $u_n := b_{n,-k_n}$ (as in Theorem 3.5). If there exists an $N \in \mathbb{N}$ such that $k_n \leq k$ for all $n \geq N$, then $u_n \to u$ as $n \to \infty$.*

Proof: Suppose there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $k_n \leq k$. From Theorem 6.33 it follows that $b_{n,-k} \to b_{-k}$ as $n \to \infty$. Therefore, since $b_{-k} \neq 0$, there exists $N_1 \in N$ such that for all $n \geq N_1$, $b_{n,-k} \neq 0$. Thus for $n \geq N_1$, $k \leq k_n$ and we have that $k_n = k$ for all $n \geq N_2 := \max\{N, N_1\}$. So for $n \geq N_2$, we have $u_n = b_{n,-k_n} = b_{n,-k}$ and $b_{n,-k} \to b_{-k} = u$ as $n \to \infty$, so that $u_n \to u$ as $n \to \infty$. □

Now we combine the above theorems and corollaries to prove the following two theorems.

Theorem 6.36 *Let (A, C) be a semisimple OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha_n \in$ psp (a_n) such that $\alpha_n \to \alpha$, then the following hold:*

1. *There exists a $N \in \mathbb{N}$ such that for all $n \geq N$, α_n is a pole, say of order k_n, of $(z - a_n)^{-1}$, and α is a pole, say of order k, of $(z - a)^{-1}$.*

Suppose, in addition, that $\inf_{n \in N} d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\}) > 0$.

2. *If*
$$(z - a)^{-1} = \sum_{j=-\infty}^{\infty} (z - \alpha)^j b_j \qquad (b_{-j} = 0 \text{ for all } j > k)$$
and for all $n \geq N$, with N as in 1,
$$(z - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (z - \alpha_n)^j b_{n,j} \qquad (b_{n,-j} = 0 \text{ for all } j > k_n)$$
then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.

3. *$p(a, \alpha) \to p(a_n, \alpha_n)$ as $n \to \infty$.*

4. *If $k_n \leq k$ for all $n \geq N_1$, for some $N_1 \in \mathbb{N}$, and $u := b_{-k}$, $u_n := b_{n,-k_n}$, then $u_n \to u$ as $n \to \infty$.*

Proof: By Theorem 2.14 psp (a) consists of Riesz points of $\sigma(a)$. Therefore, from Lemma 6.28.2, $\alpha \in$ psp (a) and hence α is a Riesz point of $\sigma(a)$ and a pole of $(z - a)^{-1}$ according to Theorem 3.11. By Theorem 6.30 α_n is a Riesz point of $\sigma(a_n)$, and hence a pole of $(z - a_n)^{-1}$, for all n big enough. This proves (1).

Using 1, we obtain 2 from Theorem 6.33, 3 from Corollary 6.34 and 4 from Corollary 6.35. □

We now come a version of the above theorem, applied to spectral radii.

Theorem 6.37 *Let (A, C) be a semisimple OBA with C closed and the spectral radius in (A, C) monotone. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. Then the following hold:*

1. *There exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $r(a_n)$ is a pole, say of order k_n, of $(z - a_n)^{-1}$, and $r(a)$ is a pole, say of order k, of $(z - a)^{-1}$.*

Suppose, in addition, that $\inf_{n \in N} d(\alpha_n, \sigma(a_n) \setminus \{\alpha_n\}) > 0$.

2. *If*
$$(z - a)^{-1} = \sum_{j=-\infty}^{\infty} (z - r(a))^j b_j \qquad (b_{-j} = 0 \text{ for all } j > k)$$

and for all $n \geq N$*, with* N *as in 1,*

$$(z - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (z - r(a_n))^j b_{n,j} \qquad (b_{n,-j} = 0 \text{ for all } j > k_n)$$

then $b_{n,j} \to b_j$ *as* $n \to \infty$*, for all* $j \in \mathbb{Z}$*.*

3. $p(a, \alpha) \to p(a_n, \alpha_n)$ *as* $n \to \infty$*.*

4. *Let* u *denote the positive Laurent eigenvector of the eigenvalue* $r(a)$ *of* a*, and* u_n *the positive Laurent eigenvector of the eigenvalue* $r(a_n)$ *of* a_n*, as in Theorem 3.5. If* $k_n \leq k$ *for all* $n \geq N_1$*, for some* $N_1 \in \mathbb{N}$*, then* $u_n \to u$ *as* $n \to \infty$*.*

Proof: The spectral radius is monotone in (A, C), so $r(a) \in \sigma(a)$ and therefore $r(a) \in \mathrm{psp}(a)$. By Theorem 2.14 $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$, so that, by Lemma 6.28.1, we have $r(a_n) \to r(a)$. The results 1-4 now follow from Theorem 6.36. $\square$

Chapter 7

Domination properties in OBA's

In this chapter we will discuss domination properties in an OBA following [7] and [17]. By a domination property in an OBA A we mean the following: If $0 \leq a \leq b$ and b has a certain property, then does a inherit this property?

First we will focus our attention to the property of being in the radical $\mathrm{Rad}(A)$ of A. We will use subharmonic analysis to get some interesting results. We get the results from subharmonic analysis that we need from [5]. Then we will look at the property of an element being inessential.

7.1 Subharmonic functions and capacity

Let D be an open subset of $\mathbb{C}$. A function ϕ from D into $\mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* on D if it is upper semicontinuous on D and satisfies the mean inequality

$$\phi(\lambda_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(\lambda_0 + re^{i\theta}) d\theta$$

for all closed disks $\overline{B}(\lambda_0, r)$ included in D. We state a few basic properties of subharmonic functions.

Let D be an open subset of the complex plane.

(i) If ϕ_1 and ϕ_2 are subharmonic on D then $\phi_1 + \phi_2$ is subharmonic on D.

(ii) If ϕ is subharmonic on D and if α is a positive number then $\alpha\phi$ is subharmonic on D.

(iii) If ϕ is subharmonic on D and if f is a real, convex and increasing function on $\mathbb{R}$, then $f \circ \phi$ is subharmonic on D (by convention $f(-\infty) = \lim f(x)$ when x goes to $-\infty$).

(iv) If (ϕ_n) is a decreasing sequence of subharmonic functions on D then $\phi = \lim_{n\to\infty} \phi_n$ is subharmonic on D.

For more properties of subharmonic functions we refer to [13] and [5].

The following theorem by E.Vesentini has a lot of applications in spectral theory and will be an important tool for the domination theorems. Therefore we will give the proof of this theorem.

Theorem 7.1 *(E. Vesentini) Let f be an analytic function on a domain D of $\mathbb{C}$ into a Banach algebra A. Then $\lambda \mapsto r(f(\lambda))$ and $\lambda \mapsto \log r(f(\lambda))$ are subharmonic on D.*

Before we can give the proof of this theorem we need the following theorem and lemma.

Theorem 7.2 *(E.F. Beckenbach-S. Saks) Let ϕ be positive on an open set D. Then $\log \phi$ is subharmonic on D if and only if $z \mapsto |e^{p(z)}|\phi(z)$ is subharmonic on D for every polynomial p.*

Proof: This is Theorem 2.6.5 in [25]. □

Lemma 7.3 *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach space X. Then $\lambda \mapsto \log \|f(\lambda)\|$ is subharmonic on D.*

Proof: This function is clearly continuous. Let $\overline{B}(\lambda_0, r)$ be a closed disk included in D. By Cauchy's theorem we have

$$f(\lambda_0) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda_0 + re^{i\theta}) d\theta,$$

and consequently

$$\|f(\lambda_0)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|f(\lambda_0 + re^{i\theta})\| d\theta. \tag{7.1}$$

For every polynomial p we have $|e^{p(\lambda)}| \cdot \|f(\lambda)\| = \|e^{p(\lambda)} f(\lambda)\|$ and $\lambda \mapsto e^{p(\lambda)} f(\lambda)$ is analytic. So by equation (7.1) applied to $e^{p(\lambda)} f(\lambda)$, $|e^{p(\lambda)}| \cdot \|f(\lambda)\|$ is subharmonic. Now we see that $\log \|f(\lambda)\|$ is subharmonic by the Beckenbach-Saks theorem. □

Now we give the proof of the Theorem 7.1:
Proof: Let $\phi_n := \frac{1}{2^n} \log \|f(\lambda)^{2^n}\|$. We have,

$$\frac{1}{2^{n+1}} \log \|f(\lambda)^{2^{n+1}}\| \leq \frac{1}{2^{n+1}} \log \|f(\lambda)^{2^n}\|^2 = \frac{1}{2^n} \log \|f(\lambda)^{2^n}\|.$$

So (ϕ_n) is a decreasing sequence. It follows from Theorem 1.10 that (ϕ_n) converges to $\log r(f(\lambda))$ as $n \to \infty$. Because $\lambda \mapsto f(\lambda)^{2^n}$ is analytic, the function $\lambda \mapsto \log \|f(\lambda)^{2^n}\|$ is subharmonic by Lemma 7.3. So $\log r(f(\lambda))$ is the limit of a decreasing sequence of harmonic functions and therefore is subharmonic by property (iv). Property (iii) now tells us that the composition of this function with e^t, which is convex and increasing, is subharmonic. Thus $r(f(\lambda))$ is subharmonic. □

To come to our main tool from subharmonic analysis that we use in domination theory, we also need the concept of capacity. We will only describe this concept vaguely and state the main theorems that we need. For more information on capacity we refer to [5] and [25]. The *capacity* c is a function from a collection of subsets of the complex plane to $\mathbb{R}$, that in some sense measures their size. For a measure, not every set is measurable; the sets that are, are called measurable sets. For capacity we also have capacitable and non-capacitable sets. The domain of c consists of the capacitable sets of the complex plane. It can be shown that the bounded subsets of the complex plane are capacitable. So for our purposes the domain of c will be the bounded subsets of $\mathbb{C}$. It can be shown that closed disks and closed line segments have a non-zero capacity. A subset of $\mathbb{C}$ is *locally of capacity zero* if all its bounded subsets have zero capacity. Therefore open disks and closed line segments are not locally of capacity zero.

Theorem 7.4 *(H. Cartan) Let ϕ be subharmonic on a domain D of $\mathbb{C}$ and not identically $-\infty$. Then $\{\lambda \in D : \phi(\lambda) = -\infty\}$ is a countable intersection of open sets which is locally of capacity zero.*

Proof: This is Theorem A.1.29 in [5]. □

We will use the following corollary of the theorem.

Corollary 7.5 *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Suppose E is either an open ball or a closed line segment with $E \subset \{\lambda \in D : r(f(\lambda)) = 0\}$. Then $r(f(\lambda)) = 0$ for all λ in D.*

Proof: If f is analytic on D, then we know that $\phi = \log(r \circ f)$ is subharmonic on D by Theorem 7.1. Suppose there is a $\lambda \in D$ with $r(f(\lambda)) \neq 0$. Then $\phi(\lambda) \neq -\infty$, so it follows from Cartan's theorem that $\{\lambda \in D : r(f(\lambda)) = 0\} = \{\lambda \in D : \phi(\lambda) = -\infty\}$ is locally of capacity zero. Since E is contained in this set, E is locally of capacity zero as well. But the assumption is that E is either an open ball or a closed line segment, which are not locally of capacity zero. So we have a contradiction. □

7.2 Domination properties

Now we turn to the domination properties in OBA's. We will use Cartan's Theorem and Corollary 7.5 to come to some interesting results. First two lemmas we are going to need:

Lemma 7.6 *Let A be an OBA such that the spectral radius is monotone. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $aC \subset \mathrm{QN}(A)$.*

Proof:

If $b \in \mathrm{Rad}(A)$, then $bC \subset \mathrm{Rad}(A) \subset \mathrm{QN}(A)$. From $0 \leq a \leq b$ and the fact that C is an algebra cone it follows that $0 \leq ac \leq bc$ for all $c \in C$. The spectral radius is monotone, because C is normal, so $r(ac) \leq r(bc)$, for all $c \in C$. Since

$bC \subset \mathrm{QN}(A)$, $r(bc) = 0$, so $r(ac) = 0$ and we have $aC \subset \mathrm{QN}(A)$. $\square$

Lemma 7.7 *Let A be an OBA. If $aC \subset \mathrm{QN}(A)$, then $a\,\mathrm{span}(C) \subset \mathrm{QN}(A)$.*

Proof: Let $n \in \mathbb{N}$ and $c_1, \cdots, c_n \in C$. Take fixed positive real numbers $\lambda_2, \cdots, \lambda_n$ and let $f_1(\lambda_1) = a(\lambda_1 c_1 + \cdots + \lambda_n c_n)$, with $\lambda_1 \in \mathbb{C}$. Then f_1 is analytic on $\mathbb{C}$. If $\lambda_1 \in \mathbb{R}^+$ then $f_1(\lambda_1) \in aC$, so by the assumption we have $r(f(\lambda_1)) = 0$ for all $\lambda_1 \in \mathbb{R}^+$. Let E be the interval $[1, 2]$, then it follows from Corollary 7.5 that $r(f_1(\lambda_1)) = 0$ for all $\lambda_1 \in \mathbb{C}$. So,

$$r(a(\lambda_1 c_1 + \cdots + \lambda_n c_n)) = 0 \quad \text{for all } \lambda_1 \in \mathbb{C} \text{ and all } \lambda_2, \cdots, \lambda_n \in \mathbb{R}^+. \tag{7.2}$$

Next we take a fixed $\lambda_1 \in \mathbb{C}$, and fixed $\lambda_3, \cdots, \lambda_n \in \mathbb{R}^+$ and let $f_2(\lambda_2) = a(\lambda_1 c_1 + \cdots + \lambda_n c_n)$, with $\lambda_2 \in \mathbb{C}$. Again f_2 is analytic on $\mathbb{C}$ and if $\lambda_2 \in \mathbb{R}^+$ it follows from equation 7.2 that $r(f_2(\lambda_2)) = 0$. Again we have by Corollary 7.5 that $r(f_2(\lambda_2)) = 0$ for all $\lambda_2 \in \mathbb{C}$, so

$$r(a(\lambda_1 c_1 + \cdots + \lambda_n c_n)) = 0 \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C} \text{ and all } \lambda_3, \cdots, \lambda_n \in \mathbb{R}^+.$$

We continue this process, until after $n - 2$ more steps we get,

$$r(a(\lambda_1 c_1 + \cdots + \lambda_n c_n)) = 0 \quad \text{for all } \lambda_1, \cdots, \lambda_n \in \mathbb{C}.$$

We took n and $c_1 \cdots, c_n \in C$ arbitrary, so $r(ax) = 0$ for all $x \in \mathrm{span}(C)$. $\square$

Since $\mathrm{Rad}(A) = \{a \in A : aA \subset \mathrm{QN}(A)\}$, we have the following theorem using Lemmas 7.6 and 7.7:

Theorem 7.8 *Let A be an OBA such that the spectral radius is monotone and suppose that $A = \mathrm{span}(C)$. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Proof: If $0 \leq a \leq b$ then by Lemma 7.6 we have $aC \subset \mathrm{QN}(A)$. Since $A = \mathrm{span}(C)$, it follows from Lemma 7.7 that $aA \subset \mathrm{QN}(A)$ and therefore $a \in \mathrm{Rad}(A)$ by Theorem 1.31. $\square$

If the span of C is dense in A we have the following result:

Theorem 7.9 *Let A be an OBA such that the spectral radius is monotone. Suppose that $A = \overline{\mathrm{span}(C)}$ and the spectral radius is continuous on A. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Proof: If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $aC \subset \mathrm{QN}(A)$ according to Lemma 7.6, so by Lemma 7.7 $a\,\mathrm{span}(C) \subset \mathrm{QN}(A)$. Because $\mathrm{span}(C)$ is dense in A, there is for each $x \in A$ a sequence $\{x_n\}$ in $\mathrm{span}(C)$ that converges to x as $n \to \infty$. Hence we have $\lim_{n\to\infty} ax_n = ax$. Each element ax_n is in $a\,\mathrm{span}(C)$, so $r(ax_n) = 0$ for all $n \in \mathbb{N}$. Since the spectral radius is continuous we have $r(ax) = \lim_{n\to\infty} r(ax_n) = 0$, i.e. $ax \in \mathrm{QN}(A)$. So $aA \subset \mathrm{QN}(A)$ and $a \in \mathrm{Rad}(A)$ by Theorem 1.31. $\square$

Using the above theorems, we can also give a characterization of the radical of A in terms of the algebra cone C:

Theorem 7.10 *Let A be an OBA such that the spectral radius is monotone and suppose that one of the following holds:*

1. $A = \mathrm{span}(C)$.

2. $A = \overline{\mathrm{span}(C)}$ *and the spectral radius function is continuous.*

Then $\mathrm{Rad}(A) = \{a \in A : aC \subset \mathrm{QN}(A)\}$.

Proof: The inclusion $\mathrm{Rad}(A) \subset \{a \in A : aC \subset \mathrm{QN}(A)\}$ is trivial, in view of Theorem 1.31.

For the other inclusion, let $a \in A$ be such that $aC \subset \mathrm{QN}(A)$. Then Lemma 7.7 tells us that $a\,\mathrm{span}(C) \subset \mathrm{QN}(A)$. In case (1), $\mathrm{span}(C) = A$, so we directly have $aA \subset \mathrm{QN}(A)$. In case (2), it follows as in the proof of Theorem 7.9 that $aA \subset \mathrm{QN}(A)$. □

Now we consider the domination problem with the property that b belongs to an ideal of A or if b is Riesz relative to some ideal of A.

Lemma 7.11 *Let (A, C) be an OBA with $a, b \in A$ and let F be a closed ideal of A. Then the following conditions are equivalent:*

1. *If $0 \leq a \leq b$ and $b \in F$, then $a \in F$.*

2. *The algebra cone πC in the quotient algebra A/F is proper.*

Proof: Suppose 1. holds. Let $\overline{c} \in \pi C \cap -\pi C$. Then $\overline{c} = \overline{c_1} = -\overline{c_2}$ for some $c_1, c_2 \in C$, so $c_1 + c_2 \in F$. We also have $0 \leq c_1 \leq c_1 + c_2$ and therefore by condition 1. it follows that $c_1 \in F$ and thus $\overline{c} = \overline{c_1} = \overline{0}$.

Suppose 2. holds. If $0 \leq a \leq b$ and $b \in F$, then $\overline{0} \leq \overline{a} \leq \overline{b} = \overline{0}$ w.r.t. πC in A/F. Since πC is proper, the order $\leq$ in A/F is antisymmetric and therefore $\overline{a} = \overline{0}$, i.e. $a \in F$. □

Lemma 7.12 *Let (A, C) be an OBA and F a closed ideal in A such that the spectral radius function in the OBA $(A/F, \pi C)$ is monotone. If $a, b \in A$ is such that $0 \leq a \leq b$ and b is Riesz relative to F, then a is Riesz relative to F.*

Proof: Let $0 \leq a \leq b$, then $\overline{0} \leq \overline{a} \leq \overline{b}$. Since the spectral radius function in $(A/F, \pi C)$ is monotone, $0 \leq r(\overline{a}) \leq r(\overline{b})$. If b is Riesz relative to F, then $r(\overline{b}) = 0$ and so $r(\overline{a}) = 0$, i.e. a is Riesz relative to F. □

Now we consider a different domination property.

Theorem 7.13 *Let A be an OBA such that the spectral radius is monotone and let $0 \leq a \leq b$ with $b \in \mathrm{QN}(A)$. If $g(a) \in \mathrm{Rad}(A)$ for some polynomial g in a with $k \in \mathbb{N}$ the smallest nonzero power of a in $g(a)$, then $a^k \in \mathrm{Rad}(A)$.*

Proof: The spectral radius is monotone, so $0 \leq r(a) \leq r(b)$. Since $b \in \mathrm{QN}(A)$ it follows that $r(a) = r(b) = 0$, i.e. $a \in \mathrm{QN}(A)$. So with the Spectral Mapping Theorem we see that $0 = \sigma(g(a)) = g(\sigma(a)) = g(0)$ and therefore $g(a) = a^k(\lambda_k + \cdots + \lambda_n a^{n-k})$ with $\lambda_k, \cdots, \lambda_n \in \mathbb{C}$, $\lambda_k \neq 0$ and $k \geq 1$. Again by using the Spectral Mapping Theorem and the fact that $a \in \mathrm{QN}(A)$, we have $\sigma(\lambda_k + \cdots + \lambda_n a^{n-k}) = \{\lambda_k\}$. Thus $\lambda_k + \cdots + \lambda_n a^{n-k}$ is invertible in A and so

$a^k = g(a)(\lambda_k + \cdots + \lambda_n a^{n-k})^{-1} \in \mathrm{Rad}(A)$. □

Now we turn our attention to inessential elements.

Theorem 7.14 *Let (A, C) be an OBA and F be a closed ideal in A . Suppose $a, b \in A$ with $0 \leq a \leq b$ and b is inessential relative to F. Let the spectral radius in the OBA $(A/F, \pi C)$ be monotone. Then:*

1. *a is Riesz relative to F.*
2. *If a is in the center of A then a is inessential relative to F.*
3. *If C is generating then a is inessential relative to F.*

Proof:

1. We already saw that $\mathrm{kh}(F) \subset R(J)$. So if b is inessential relative to F, then b is Riesz relative to F. Also, the spectral radius in the quotient algebra A/F is monotone and the result follows from Theorem 7.12.
2. Since b is inessential relative to F, we have $\overline{b} \in \mathrm{Rad}(A/F)$. We suppose that a is in the center of A, so $\overline{a}$ is in the center of A/F. Now it follows from Theorem 2.3.3 that $\overline{a} \in \mathrm{Rad}(A/F)$, thus a is inessential relative to F.
3. Since C is generating in A, πC is generating in the quotient algebra A/F. It follows from Theorem 7.10 that $\overline{a} \in \mathrm{Rad}(A/F)$.

□

From this theorem we get the following corollaries

Corollary 7.15 *Let (A, C) be an OBA and F a closed ideal in A. Suppose the spectral radius in the OBA $(A/F, \pi C)$ is monotone and C is generating. Then the algebra cone $C + \mathrm{kh}(F)$ in the quotient algebra $(A/\mathrm{kh}(F), C + \mathrm{kh}(F))$ is proper.*

Proof: Let $0 \leq a \leq b$ and $b \in \mathrm{kh}(F)$, then Theorem 7.14.3 tells us that $a \in \mathrm{kh}(F)$. Because $\mathrm{kh}(F)$ is a closed ideal it follows from Theorem 7.11 that the algebra cone $C + \mathrm{kh}(F)$ in the quotient algebra $(A/\mathrm{kh}(F), C + \mathrm{kh}(F))$ is proper. □

Corollary 7.16 *Let (A, C) be an OBA and F a closed ideal in A such that $\mathrm{kh}(F)$ is a proper ideal in A. Suppose $a, b \in A$ with $0 \leq a \leq b$ and b inessential relative to F. If the spectral radius in the OBA $(A/F, \pi C)$ is monotone and C is generating, then a cannot be invertible.*

Proof: Using the fact that a proper ideal cannot contain invertible elements it follows from Theorem 7.14.3. □

Theorem 7.13 in the case of inessential elements becomes:

Theorem 7.17 *Let (A, C) be an OBA and F a closed ideal in A such the spectral radius in the OBA $(A/F, \pi C)$ is monotone. Let $a, b \in A$ such that $0 \leq a \leq b$ and let b be Riesz relative to F. If $g(a)$ is inessential relative to F for some polynomial g in a with $k \in \mathbb{N}$ the smallest nonzero power of a in $g(a)$, then a^k is inessential relative to F.*

Proof: Since $0 \leq a \leq b$, we have $0 \leq \overline{a} \leq \overline{b}$. The element b is Riesz relative to F, so $b \in \mathrm{QN}(A/F)$. The spectral radius in the *OBA* $(A/F, \pi C)$ is monotone and if $g(a)$ is inessential relative to F, then $g(a) \in \mathrm{Rad}(A/F)$. Therefore we can apply Theorem 7.13 to the *OBA* $(A/F, \pi C)$ and we see that $a^k \in \mathrm{Rad}(A/F)$, i.e. a^k is inessential relative to F. $\square$

Theorem 7.18 *Let (A, C) be an OBA with C closed and the spectral radius in (A, C) monotone. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a, b \in A$ with $0 \leq a \leq b$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$, then psp(a) consists of Riesz points of $\sigma(a)$.*

Proof: Since $r(b)$ is a Riesz point of $\sigma(b)$, it follows from lemma 2.12.1 that $r(\overline{b}) < r(b)$. By the monotonicity of the spectral radius in $(A/I, \pi C)$ we have that $r(\overline{a}) \leq r(\overline{b})$, and, since $r(a) = r(b)$, it follows that $r(\overline{a}) < r(a)$. Lemma 2.12.2 implies that $r(a)$ is a Riesz point of $\sigma(a)$. The result now follows from Theorem 2.14. $\square$

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